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STATISTICAL TABLES

FOR BIOLOGICAL, AGRICULTURAL AND MEDICAL RESEARCH

BY

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PREFACE

THE problems with which an active statistical department may have to deal require, if their solutions are to be made widely accessible, a great variety of special tables. A number of those in the present book are familiar to statisticians, and are already widely used. In presenting them in a convenient form, the opportunity has been taken to supplement them with a selection of others, chosen as likely also to be of value, and not accessible elsewhere. The volume is completed with a number of tables of standard functions of general utility. The experience of the authors of the problems arising in practical research, is the basis of the selection, from among those tables which from time to time have been computed for special purposes.

To perform their compilation, the main work has been to complete the tables in a form suitable for the general user, and, what is by no means easy, to ensure their entire accuracy. In addition we have been principally concerned with the design of printing and arrangement, which shall be suitable both for constant and for occasional users. Comments from both classes of reader will therefore be especially valued.

We are indebted to all workers at the Galton Laboratory, and at the Statistical Laboratory, Rothamsted, and especially to W. L. Stevens, for assistance in the completion of our task.

R. A. F.
F. Y.

August 1938

PREFACE TO THIRD EDITION

THE second edition contained as new material Tables V1 and V2 for the test of significance of the differences between two means obtained by different methods; Table VIII1, which supplies solutions of the type of problem illustrated in Examples 1, 2 and 4, more direct than those previously available; and Table VIII2, which gives densities of organisms estimated by the dilution method. Additions to the third edition consist of Table XI1 (due to Mr D. J. Finney) which gives the modified probit weighting coefficients necessary for dosage mortality tests involving deaths among controls; the inclusion of the 10 per cent. points of z and e^{2z} (based on the tables of the

incomplete Beta function of Miss Catherine M. Thompson, for which we are indebted to Prof. E. S. Pearson and Dr V. G. Panse) in Table V; the extension of Table XXIII of orthogonal polynomials for values of n' from 53 to 75 (due to Mr V. Satakopan); and the extension of Table XXVI of natural logarithms to cover the range ·100 to 1·00.

The combinatorial solutions of Table XVII were considerably extended and simplified in presentation in the second edition by the more extensive use of solutions of the cyclic type, and the corresponding section of the introduction was expanded to give an account of Youden's Squares, and of the method of utilising information, formerly discarded, from comparisons between blocks. Further new solutions have been added in the third edition.

Table VIII has been improved in the third edition so as to allow for interpolation for values of the observed event above 10; Table XI, of weighting coefficients and probit values for adjustments of special accuracy, has been recast in order to simplify its use; Table XXXIV of constants, weights and measures, etc., has been slightly revised; and a small positive bias has been removed from the 1 per cent. values of Table VI for the significance of the difference between two means.

For the convenience of owners of the second edition, a list of errata has been included showing all the errors of any importance that have been discovered in this edition. Our thanks are due to those who have drawn our attention to these errors, and also to Mr M. H. Quenouille and to Dr Rose O. Cashen for their assistance in preparing the third edition for the press.

R. A. F.
F. Y.

January 1947

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ERRATA TO THE SECOND EDITION

The page numbers are those of the second edition; all these errors have been corrected in the third edition.

p. 8. *For* 1·976 *read* 1·960.
p. 37. $n_1 = 8$, $n_2 = 5$. *For* 10·27 *read* 10·29.
p. 48. Table X, 1/9. *For* 1·2263 *read* 1·2206.

INTRODUCTION

Tables I to VII. Tests of Significance based on the Normal Distribution

Tables I to VII constitute a group of tables, based on the normal distribution, and now widely used in making tests of significance. The common uses of these tables are fully illustrated with numerical examples in Fisher's *Statistical Methods for Research Workers*, where they were first published. Further statistical problems soluble by the same tables are, however, constantly being discovered.

If x is the deviation from the mean of a normal distribution having unit variance, the ordinate z, given in Table II, is obtained from the algebraic expression

$$z = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

The integral of this function gives the area of the curve, or the frequency of observations, between the limits of integration. The fraction of the frequency falling outside the limits $\pm x$ is, therefore,

$$P = 2\int_x^\infty z\,dx.$$

Table I gives the values of x corresponding with various values of this fraction.

The distribution of t, due originally to "Student," is used to test the significance of a deviation when its standard error is *estimated from the data*; t is the deviation divided by its estimated standard error. The algebraic expression for the ordinate of the distribution is

$$\frac{\left(\frac{n-1}{2}\right)!}{\left(\frac{n-2}{2}\right)!\sqrt{\pi n}} \cdot \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{1}{2}(n+1)}}.$$

Table III gives the values of t corresponding with various values of the probability of a random value falling outside the limits $\pm t$. The integer n is the number of degrees of freedom available for the estimation of error.

The quantity χ^2, used to test goodness of fit, homogeneity, and for a variety of other purposes, may be regarded as the sum of the squares of n variables, which vary normally and independently about zero with unit variance. It is a useful single measure of the aggregate of a set of deviations from expectation. The element of frequency may be expressed as

$$df = \frac{1}{\left(\frac{n-2}{2}\right)!}\left(\tfrac{1}{2}\chi^2\right)^{\frac{1}{2}(n-2)} e^{-\frac{1}{2}\chi^2}\, d\left(\tfrac{1}{2}\chi^2\right).$$

When n is even the probability of exceeding any given value χ^2 is reducible to the partial sum of a Poisson series:

$$P = \int_{\chi^2}^\infty df = e^{-\frac{1}{2}\chi^2}\left\{1+\left(\tfrac{1}{2}\chi^2\right)+\frac{1}{2!}\left(\tfrac{1}{2}\chi^2\right)^2+\ \ldots\ +\frac{1}{\left(\frac{n-2}{2}\right)!}\left(\tfrac{1}{2}\chi^2\right)^{\frac{1}{2}(n-2)}\right\}.$$

The mean of the whole Poisson series is $\frac{1}{2}\chi^2$, the sum of the whole series is unity, and the number of terms in the partial sum is $\frac{1}{2}n$. In consequence the table of χ^2 (Table IV), which gives the values of χ^2 for various values of P, can be used to find the mean of a Poisson series of which the first $\frac{1}{2}n$ terms constitute a given fraction of the whole.

Example 1

Four serious floods have occurred in a given region during a century's experience; supposing the numbers of floods experienced in different centuries are distributed in a Poisson series, find the mean number for which only one century in 20 would have four serious floods or less, and the mean number for which only one century in 20 would have 4 floods or more.

In the first case we need the first five terms of the Poisson series, so $n = 10$; the table shows the 5 per cent. value of χ^2 is 18·307; the expectation would be 9·154 floods per century. If, on the contrary, only one century in 20 experienced 4 floods or more, the first four terms of the series would constitute 95 per cent. of the whole;

for $n=8$, the 95 per cent. χ^2 is 2·733, corresponding with a frequency of 1·366 serious floods per century. Unless our experience has fallen in the upper, or the lower 5 per cent. range of the distribution, the expectation lies between 1·366 and 9·154 floods per century. More or less stringent probabilities might, of course, have been chosen. Table VIII1 has been constructed to make this test directly at the 10 per cent., 2½ per cent. and ½ per cent. limits. Entering the table with $a = 4$ and $N = \infty$, for instance, the 2½ per cent. limits of the expectation are found to be 1·09 and 10·24.

In testing goodness of fit χ^2 is simply calculated from the numbers observed and expected in each distinguishable class. If m is the frequency expected and a that observed, then

$$\chi^2 = S\frac{(a-m)^2}{m} = S\left(\frac{a^2}{m}\right) - S(a)$$

where the summation is taken over all classes. The number of degrees of freedom is the number of classes, the frequencies of which can be varied independently without violating any totals, sub-totals, etc., which may have been used in calculating the expectations. It is inadvisable to use classes with expectations less than 5, so that these will usually be grouped in larger classes in applying the test. (See, however, Table VIII.)

The quantity z may be defined as the difference of one-half the natural logarithms of two different estimates of variance, one based on n_1 and the other on n_2 degrees of freedom. Table V gives values of z for different values of the three variables n_1, n_2 and P. The corresponding values of the variance ratio, e^{2z}, are given on the opposite pages. We may usually consider a z test as a test whether one estimate of a variance based on n_1 degrees of freedom significantly exceeds a second estimate based on n_2 degrees of freedom. We are indebted to C. G. Colcord and L. S. Deming (9) for the extension of the z table to $P = \cdot001$, to H. W. Norton and Iowa State College for $P = \cdot2$, and to Catherine M. Thompson (40, 41) and the Department of Statistics, University College, London, and to V. G. Panse (42) for $P = \cdot1$.

The distributions of t and of χ^2 are equivalent to special cases of the distribution of z. Thus when $n_1 = 1$, $t = e^z$ and t^2 for n_2 degrees of freedom is the variance ratio. Equally, when $n_2 = 1$, the variance ratio is the value of $1/t^2$ corresponding with the probability $1-P$.

When $n_2 = \infty$, the variance ratio is χ^2/n_1 for n_1 degrees of freedom; and when $n_1 = \infty$ it is the value of n_2/χ^2 for n_2 degrees of freedom, corresponding with the probability $1-P$.

From the value 6 onwards the values of n_1 tabulated are in harmonic progression; *i.e.* they are the numbers for which $24/n_1$ takes the values 4, 3, 2, 1 and 0. As the corresponding values of z run smoothly other values of n_1 may be used by interpolating for the values of $24/n_1$. Similarly, the largest n_2 tabulated are those for which $120/n_2$ takes the values 4, 3, 2, 1 and 0. The z values for any pair of numbers n_1 and n_2 can thus be found from the table. With lower accuracy the variance ratio may be similarly interpolated. For high values of n_1 and n_2 a useful approximation is given below the table for each value of P tabulated.

In the table of z, like that of χ^2, the probability given, and most frequently wanted, is that of a single tail of the distribution beyond the tabulated value. For the tables of x and t it is the sum of two equivalent tails.

In addition to its general use in the analysis of variance, the z table is related to the partial sum of a binomial series in the same way as is that of χ^2 to the Poisson series. The ratio of the sums of squares is n_1/n_2 times the variance ratio, or n_1e^{2z}/n_2. If $p : q$ is the ratio of the sums of squares, P is the sum of the first $\frac{1}{2}n_1$ terms of the expansion of $(q+p)^{\frac{1}{2}(n_1+n_2-2)}$, and $1-P$ is the sum of the remaining $\frac{1}{2}n_2$ terms.

Example 2

An event has occurred 3 times out of 14 trials. The probability of its occurring 3 times or less, is represented by the sum of the first four terms out of 15, hence $n_1 = 8$, $n_2 = 22$. If $P = 1$ per cent., the table gives a variance ratio 3·45; hence the ratio of the sums of squares is $3\cdot45\times8/22 = 1\cdot255 = p/q$. Thus $p = 1\cdot255/2\cdot255 = \cdot557$. On the other hand, for the probability of its occurring 3 times or more, we need $n_1 = 24$, $n_2 = 6$, when the variance ratio is 7·31, and $q/p = 29\cdot24$, $p = 1/30\cdot24 = \cdot0331$. Unless our experience has fallen in the upper or the lower 1 per cent. range of the distribution, the probability of occurrence lies between ·0331 and ·557.

This type of problem also may be examined at other levels of significance by the use of Table VIII1. Entering the table with $a = 3$, $p = 3/14 = 0\cdot214$, we find that the limits of expectation, at the probability level $P = \frac{1}{2}$ per cent., are 0·360 and 8·26, giving the corresponding limits of the probability of occurrence as ·0257 and ·590. If $P = 2\frac{1}{2}$ per cent. the limits are ·0466 and ·508.

If the event had not occurred at all in 14 trials, it is still possible to assign an upper limit to the probability. For $P = 2\frac{1}{2}$ per cent., for example, the upper limit to the probability is $1-\cdot025^{\frac{1}{14}}$ or ·232, and for $P = \frac{1}{2}$ per cent. it is $1-\cdot005^{\frac{1}{14}}$ or ·315, corresponding to upper limits of expectation of 3·25 and 4·41 respectively. The same result can be obtained by interpolation with respect to $1/N$ in Table VIII1.

Table VI gives the exact test of significance for inter-class correlation coefficients. This is equivalent to the t test of the regression, which reduces to

$$t = \frac{r}{\sqrt{1-r^2}}\sqrt{n'-2}$$

for $n'-2$ degrees of freedom, where n' is the number of pairs on which the correlation is based. The results often differ materially from the comparison of r with its standard error estimated from the formula,

$$\sigma_r = \frac{1-r^2}{\sqrt{n'-1}}.$$

Other comparisons involving correlation coefficients are best carried out by the z transformation provided by Table VII.

Tables V1 and V2. Significance of Difference between Two Means

Tables V1 and V2 give the test of significance, due originally to Behrens, for the difference of two means for which the errors are due to different causes, so that estimates of variance cannot properly be pooled. In general we need the percentile points of a distribution compounded of two Student distributions with n_1 and n_2 degrees of freedom. The ratio of the scale-factors s_1 and s_2 is given by

$$s_1/s_2 = \tan\theta,$$

and the difference is judged to be significant if it exceeds $d\sqrt{s_1^2+s_2^2}$.

For each level of significance the value of d is required given the 3 known values n_1, n_2 and θ. Table V1 by P. V. Sukhatme gives the 5 per cent. and 1 per cent. points for 15° intervals of θ, and for the lattice of values 6, 8, 12, 24 and ∞ for n_1 and n_2. Four-point interpolation is sufficient for full accuracy.

Table V2 deals more fully with the marginal case $n_1 = \infty$, where the distribution is compounded of a normal component of known variance and a component of Student's type. This occurs when we compare estimates of a physical constant, one of which is based on a large quantity of data of low precision, while the other rests on few, but rather precise, observations. When the errors from these two estimates are of comparable magnitude, both can be taken into account by finding the angle, defined by

$$\tan\theta = \sigma/s,$$

while the significant value for the observed difference is

$$d\sqrt{s^2+\sigma^2}.$$

Thus for high angles the distribution tends to normality and for low angles it tends to Student's form.

Sukhatme's values for 1 per cent. have been revised to remove the very small positive bias, about ·001 in d, detected by comparison with the asymptotic calculation used in constructing Table V2.

Example :—

Let us suppose that a physical constant evaluated by a new method gives a mean of twelve determinations,

$$\bar{x} = 4{\cdot}77383,$$

and that the sum of the squares of the deviations of these values from their mean is

$$S(x-\bar{x})^2 = {\cdot}011580,$$

so that from 11 degrees of freedom the variance of the mean is estimated to be

$$s^2 = {\cdot}00008773,$$

and the estimated standard deviation of the mean is

$$s = {\cdot}009366.$$

For 11 degrees of freedom Student's t is 2·718 at the 2 per cent. point, so that at this level of significance we should regard our observed mean as differing significantly from any value outside the limits

$$\bar{x} \pm ts$$
$$= 4{\cdot}77383 \pm {\cdot}02546$$
$$= 4{\cdot}74837,\ 4{\cdot}79929.$$

If, now, numerous previous determinations, using a different method, have given the value,

$$\mu = 4{\cdot}744,$$

outside these limits, it would appear that there must be a real discrepancy between the results from the two methods used.

The comparison, however, is not complete, if we realise that μ also is not a perfectly precise determination, for it might be that though the value adopted is incompatible with the new observations, yet that both methods might be simultaneously compatible with some intermediate value of the constant.

Behrens' test enables this question to be answered, when the precision of μ is known, by averaging the

probability of obtaining so high a deviation, when μ is assigned a frequency distribution instead of a fixed value. If, for example, the value of μ adopted, has a standard error, σ, based on a large number of degrees of freedom, with

$$\sigma = \cdot 00382,$$

we should find,

$$\tan\theta = \sigma/s = \cdot 4079,$$
$$\theta = 22^\circ\ 12';$$

and

$$d = \frac{\bar{x}-\mu}{\sqrt{s^2+\sigma^2}} = \frac{\cdot 02983}{\cdot 010115} = 2\cdot 949$$

Table V2 shows that at the 2 per cent. level of significance d is only 2·704 at 20° and 2·637 at 30°, for 10 degrees of freedom. For 11 degrees of freedom lower values still would be significant. The discrepancy observed is therefore a significant one, even when the errors to which μ is liable are fully taken into account.

Table VIII. Tests of Significance for 2×2 Contingency Tables

Contingency tables are normally tested for significant departures from independence by means of the χ^2 test. Provided that more than one degree of freedom is involved this test is satisfactory, even when the expectation in each class is quite small ; but with only a single degree of freedom (as in 2×2 contingency tables) the number of possible sets of values satisfying the given marginal totals is small, so that the discontinuity of the resultant distribution is marked, whereas the χ^2 distribution, which is an approximation to it, is continuous. The effect of this discontinuity can be overcome by applying Yates's correction for continuity when calculating χ^2. This is done by reducing by $\frac{1}{2}$ the values which are greater than expectation and increasing by $\frac{1}{2}$ those which are less than expectation.

The correction for continuity, which should be applied when the smallest expectation is less than 500, considerably increases the effective range of the ordinary χ^2 test for a single degree of freedom ; but when the smallest expectation is less than 100, the possible asymmetry of the actual distribution associated with given marginal totals becomes of importance. Such asymmetry implies that deviations of a given magnitude in one direction are less probable than those in the other.

Table VIII allows this asymmetry (and other slight deviations from the ordinary χ^2 distribution) to be taken into account. It gives what are in effect the 2·5 per cent. and 0·5 per cent. points of χ_c (the square root of χ^2 corrected for continuity) for each tail separately. Since there are three independent marginal totals in a 2×2 contingency table all possible contingency tables could only be completely covered by a three-dimensional table of χ_c for each level of significance. In general, however, the distribution of χ_c is not very different for sets of marginal totals which give the same smallest expectation, and for which the observed value in any one cell can assume the same number of values (*i.e.* tables in which the smallest marginal total is the same, this being one less than the possible number of values).

This property is utilised in Table VIII, where values of χ_c are shown for chosen values of the smallest expectation m, and of p, the smallest expectation divided by the smallest marginal total. (For a detailed account of the method of construction of this table see Yates (14).)

In the contingency table :—

x	y	a'
z	u	b'
a	b	n

where $a \leqslant b$, $a' \leqslant b'$ and $a \leqslant a'$, so that a is the smallest marginal total, and x is the number in the cell with smallest expectation, we have $m = aa'/n$, and $p = m/a = a'/n$. It will be noted that as n increases with fixed m and p, a remains fixed and $a' : b' : n$ is constant. When n is very large the distribution of x tends to that of the binomial distribution $(p+q)^a$. When n takes on its smallest possible value, which is that which gives the smallest integral value of $a' \geqslant a$, the " limiting contingency distribution " of x is attained. The values of χ_c for these two extreme distributions are shown in Table VIII, the former in ordinary type and the latter in black type. In general the intermediate contingency distributions give intermediate values of χ_c.

Furthermore, as p tends to zero with fixed m, the expectations in all cells except that containing x become large, and the Poisson distribution is approached, whatever the ratio of a to a'. Thus for $p = o$ there is only one set of values in Table VIII, representing the virtual 2·5 and 0·5 per cent. points of the Poisson distributions. Thus the table can be used to test the significance of departures from binomial distributions with known p and Poisson distributions with known m, as well as for contingency tables.

Example 3

The number of summers (April-September) with periods of over 20 days without rainfall at Rothamsted, during this century and the last half of last century, are given in Table 1.

TABLE 1. SUMMERS WITH AND WITHOUT DROUGHT AT ROTHAMSTED

	1852-1900.	1901-1937.	Total.
Experiencing drought . .	4	9	13
Without drought . . .	45	28	73
Total . . .	49	37	86

Are the summers of the present century significantly more subject to droughts than those of the last?

$$\text{Here} \quad \chi^2 = \frac{(9\times45-4\times28)^2\times86}{49\times37\times13\times73} = 4\cdot291,$$

which would indicate significance, the 5 per cent. point for one degree of freedom (Table IV) being 3·841. Applying the correction for continuity, however, we obtain

$$\chi_c^2 = \frac{(8\frac{1}{2}\times44\frac{1}{2}-4\frac{1}{2}\times28\frac{1}{2})^2\times86}{49\times37\times13\times73} = 3\cdot124.$$

Reference to Table IV indicates that significance is not attained. This may be confirmed by reference to Table VIII. We have

$$\chi_c = 1\cdot77, \text{ the smallest expectation } m = \frac{13\times37}{86} = 5\cdot59, p = \frac{m}{13} = \frac{37}{86} = 0\cdot43.$$

The observed set lies on the longer tail of the distribution since the number 9 with smallest expectation is greater than expectation. All the values of Table VIII ($P = \cdot025$) which bracket the required value lie between 1·91 and 2·06, so that significance is not attained. Notice, however, that had p been nearly zero, with the same expectation in the smallest class but with observed set on the shorter tail, the observed value of χ_c would have indicated significance.

Linear interpolation in two dimensions in Table VIII gives the value of 1·96 for both binomial and limiting contingency distributions, indicating that at this particular point the ordinary χ^2 table (or a table of the normal probability integral, *e.g.* the table of probits, Table IX) gives a good estimate of the actual probability after the correction for continuity has been applied. The probit corresponding to 1·77 is 6·77, which gives a percentage of 96·16 on inverse interpolation in Table IX, corresponding to a probability of ·0384 compared with the true value of ·0389 calculated by the procedure given in *Statistical Methods*, Section 21.02. A method of assessing the actual probability when the values of χ_c differ from those of χ for the same probability levels is illustrated in Example 5.

Example 4

In the course of 500 spins of a roulette wheel, zero appears 24 times. What evidence is there that the wheel is biased in favour of the bank?

If the wheel is unbiased, the chance of zero occurring in any one spin is $\frac{1}{37}$, and zero may be expected 13·514 times on the average in 500 trials. The discrepancy is therefore 10·486, $\chi^2 = \frac{10\cdot486^2}{13\cdot514} + \frac{10\cdot486^2}{486\cdot486}$ $= 8\cdot362$, and $\chi = 2\cdot892$. (Note that the class of non-occurrences as well as the class of occurrences must be included when calculating χ^2.) Inverse interpolation in Table IX gives a percentage of 99·809, which indicates that the probability of obtaining 24 or more zeros was only ·00191. Correcting for continuity, the discrepancy becomes 9·986, and $\chi_c^2 = 7\cdot584$, $\chi_c = 2\cdot754$, giving a probability of ·00294. Referring to Table VIII, however, for $m = 13\cdot5$ and $p = \cdot027$ (values in ordinary type) we see that the 0·5 per cent. point of χ_c is about 2·80, compared with the value of 2·576 given by normal theory. Thus the true probability is somewhat greater than ·005. (True value obtained by summing the binomial distribution = ·00559.) The omission of the correction for continuity introduces large errors, and even the use of the ordinary χ^2 table gives a probability of one half the true value.

The problem may also be solved directly by the use of Table VIII1. Since a is greater than 10 we must use the corrections given for $a = 11$ and over. $p = 24/500 = \cdot048$ and therefore the conventional standard error $\sqrt{\{a(1-p)\}} = 4\cdot79$. Multiplying this by the normal deviate for $P = \frac{1}{2}$ per cent. (single tail) gives $4\cdot79 \times 2\cdot576 = 12\cdot34$. The limits of expectation calculated from the conventional standard error are therefore $11\cdot66$ and $36\cdot34$. The corrections given by Table VIII1 for these limits are $+1\cdot70$ and $+2\cdot80$, giving the adjusted limits of $13\cdot36$ and $39\cdot14$, corresponding to probability limits $\cdot0267$ and $\cdot0783$. Since $\frac{1}{37} = \cdot0270$ the probability of obtaining as many as 24 zeros in 500 spins is just over $\cdot005$, agreeing with our previous solution.

Example 5

Locate the upper and lower $2\cdot5$ and $0\cdot5$ per cent. points of a Poisson distribution with an expectation of 25.

If x is the observed value and this is greater than expectation (*i.e.* lies on the longer tail) we have $\chi_c = (x-25-\frac{1}{2})/\sqrt{25}$. For $m = 25$, $p = 0$, $\chi_c = 2\cdot05$ for the $2\cdot5$ per cent. point, so that $x = 35\cdot8$. Hence a value of 36 or over has a smaller probability than $\cdot025$. For the $0\cdot5$ per cent. point, $\chi_c = 2\cdot76$, so that $x = 39\cdot3$, giving a value of 40 or over. Similarly, the lower $2\cdot5$ and $0\cdot5$ per cent. values of x will be found to be $15\cdot2$ and $12\cdot6$, giving integral values of 15 and 12 respectively. Normal theory would give x's about $\frac{1}{2}$ a unit in error for the $2\cdot5$ per cent. points and about a unit in error for the $0\cdot5$ per cent. points.

If the actual value of the probability of obtaining a value of say 14 or less is required, we may proceed as follows. At the lower $2\cdot5$ per cent. point χ_c is $1\cdot86$ and χ (normal theory) is $1\cdot96$, and at the lower $0\cdot5$ per cent. point χ_c is $2\cdot37$ and χ is $2\cdot58$. A value of 14 gives a χ_c of $2\cdot10$. Hence by linear interpolation, using χ_c as argument, the corresponding value of χ is

$$\chi = 1\cdot96 + \frac{2\cdot10-1\cdot86}{2\cdot37-1\cdot86}(2\cdot58-1\cdot96) = 2\cdot25.$$

Reference to Table IX gives the corresponding probability as $\cdot0122$. (True value $= \cdot0124$.)

Table VIII1. Limits for the Probability of an Event

If an event is observed to occur a times out of N, a lower limit π_1 can be assigned to the probability of this event such that if the probability were actually π_1 then an observed number of occurrences as great or greater than a out of N trials would only occur by chance with a frequency of P. Similarly an upper limit π_2 can be assigned such that if the probability were actually π_2 an observed number of occurrences as small or smaller than a would occur with a frequency P. Corresponding to these probabilities π_1 and π_2 there are limits of expectation of the number of occurrences in N trials, namely $\pi_1 N$ and $\pi_2 N$. These limits of expectation are tabulated in Table VIII1 for $P = 0\cdot1$, $0\cdot025$ and $0\cdot005$ against values of a from 0 to 10 (with extension to higher values) and of N from $2a$ to ∞. The values for $N = \infty$ gives the limits of expectation of the Poisson distribution. When $a > \frac{1}{2}N$, the table may be entered with $a' = N-a$. Examples of the use of the table have already been given (Examples 1, 2 and 4).

Table VIII2. Densities of Organisms Estimated by the Dilution Method

One of the methods available for estimating the number of organisms of a given type present in a medium is the dilution method. This method can be employed in cases in which the presence or absence of the organism can be determined, but in which counts cannot be made.

In the dilution method a series of suspensions of the organism, s levels in all, is prepared, each of which is a times as dilute as the preceding one. Each suspension is used to inoculate n tubes with a known volume of the suspension, and the presence or absence of the organism is determined by suitable test after a period of incubation. The most accurate estimate of the number of organisms per tube at any given level is obtained by solving the equation of maximum likelihood, but it has been shown (Fisher, 30) that $87\cdot7$ per cent. of the information is contained in the total number X of fertile or Y of sterile plates, counted without regard to level. If λ is the number of organisms per tube at the highest concentration, the value of λ for which the expected average number of sterile plates is equal to the observed number is given by the equation

$$Y = n(e^{-\lambda} + e^{-\lambda/a} + e^{-\lambda/a^2} + \ldots + e^{-\lambda/a^{s-1}})$$

Table VIII2 enables the solution of this equation to be obtained expeditiously for two-fold, four-fold or ten-fold dilution series of any length. All that is necessary is to calculate what may be called the mean fertile level, $x = X/n$, and the mean sterile level, $y = s-x$, and find the tabular value K corresponding to the observed x or y for the appropriate number, s, of levels. The estimate of λ is then given by

$$\log \lambda = x \log a - K.$$

In an extended series of dilutions the value of K becomes constant for all values of x and y sufficiently distant from the ends of the distribution, except for a periodic component, which, however, is only of importance for the ten-fold series. This periodic component necessitates the special form of tabulation adopted for this series.

The average value of the variance of the mean fertile level is $\frac{1}{n}\frac{\log 2}{\log a}$ except at the ends of the series, where the value increases. The average value of the variance of $\log \lambda$ is $\frac{1}{n} \log 2 \,.\, \log a$, and this may be used in the two and four-fold series in the range where the tabular entries are substantially constant. The variance of the mean fertile level is, however, more stable than the variance of $\log \lambda$. Closer approximation to the fiducial limits can therefore be found by using the average value $\frac{1}{n}\frac{\log 2}{\log a}$ of this variance, increasing and decreasing the observed mean fertile level by the appropriate multiple of the corresponding standard error. The table can then be re-entered to determine limiting values of $\log \lambda$, and thence, if required, of λ. This approximation is satisfactory except at the extreme ends of the range. For the same reason the mean fertile level should be used in preference to $\log \lambda$ when performing tests of significance, etc., on comparisons between different media. Standard errors should not be attached to λ.

The inclusion of tables for the four-fold and ten-fold series should not be taken to imply that the use of these series is ordinarily advisable. If an adequate number of tubes is available the two-fold dilution series should be used, with correspondingly fewer plates at each level, in preference to a four-fold or ten-fold series covering the same range. As is shown by the formula given above, the average variance of $\log \lambda$ in the centre of the range is the same for a given total number of plates, whatever the dilution factor, but the periodicity becomes appreciable for the more coarsely spaced series.

The number of dilution levels required depends primarily on the range of populations it is desired to cover. For any given population density the majority of the information is given by the levels at which the average number of organisms per tube is near 1. Thus in a two-fold dilution series in which one dilution level has an average of exactly one organism per tube the contribution of the various levels of an infinite series to the total amount of information are as follows :

Average number of organisms per tube .	8 and more	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$ and less
Percentage of information	0·9	12·6	26·4	24·5	16·2	9·3	4·9	5·2

For any given population density, therefore, most of the information is contributed by five levels of the series. A series of 12 dilutions will therefore amply cover a range of 100 fold, and a series of 15 dilutions a range of 1000 fold. If more than one organism which can be detected in the same set of tubes is under investigation, then longer series may be required, the exact length depending on the relative numbers of the different types of organism.

When a large number of tests involving series of the same length and with the same number of tubes at each level have to be made, the experimenter may find it convenient to prepare a special table giving $\log \lambda$ and λ directly for each possible value of the total number of fertile tubes. Such a table may be easily prepared from Table VIII2.

In practical work greater precision is often required in one part of the range than in the others. Thus the tests may have as their primary object the determination of whether the population densities are less than a given standard density, but at the same time a rough estimate of the actual population densities of samples differing widely from the standard may be of interest. In such a case the number of tubes should be increased at the levels for which the population per tube would be in the neighbourhood of 1 at the standard density, *e.g.* we might increase the number of tubes at levels having 4, 2, 1, $\frac{1}{2}$ and $\frac{1}{4}$ organisms per tube at the standard density.

For the estimation of the population densities of dilution series of this type a special table will have to be prepared. If there are n_1 tubes at each of s levels and an additional n_2 tubes at each of the levels from $r+1$ to r', the average total number of sterile tubes is given by

$$Y = n_1 y_1 + n_2 y_2 = n_1(e^{-\lambda} + e^{-\lambda/2} + \ldots + e^{-\lambda/2^{s-1}}) + n_2(e^{-\lambda/2^r} + \ldots + e^{-\lambda/2^{r'-1}})$$

The table can be prepared directly by calculating the values of Y for a series of values of $\log \lambda$, thence determining by inverse interpolation the values of $\log \lambda$ corresponding to integral values of Y. Alternatively Table VIII 2 may be used to determine the values of y_1 and y_2 separately for a series of values of $\log \lambda$, thence performing the inverse interpolation as before. Values of y_2 corresponding to values of $\log \lambda$ falling outside the range of the table will in any case have to be computed directly, but this can be done rapidly, using the expansion of y_2 in powers of λ for small values of λ.

The standard errors of estimates derived from a modified dilution series of this type naturally vary in different parts of the range. At the standard density, however, the standard errors will be those corresponding

to an infinite series of n_1+n_2 tubes at each level, except for a slight increase due to the shortness of the subsidiary series.

Example 5. 1

Tests with potato flour containing rope spores (B. mesentericus) gave the following observations, using 5 tubes, each of 1 c.c., of dilutions containing 4, 2, 1, . . ., 1/128 g. per 100 c.c. (E. C. Barton-Wright's data).

g. per 100 c.c.	Number fertile.	g. per 100 c.c.	Number fertile.
4	5	1/8	3
2	5	1/16	2
1	5	1/32	2
1/2	5	1/64	0
1/4	4	1/128	0

The total number of fertile plates is 45 and the mean fertile level (x) is therefore 6·2. Hence y is 3·8. The table gives $K = \cdot 383$. Hence

$$\log \lambda = 6\cdot 2 \log 2 - \cdot 383 = 1\cdot 483$$
$$\lambda = 30\cdot 4.$$

The number of organisms per gram is thus estimated to be 760.

The expected variance of the mean fertile level is 0·2. The fiducial limits of the number of tubes at levels ·025 to ·975 for F are therefore $6\cdot 2 \pm 1\cdot 960 \times \sqrt{0\cdot 2} = 5\cdot 323$ and 7·077. Re-entering the table with these values, we obtain limits of 1·211 and 1·761 for log λ, or 407 and 1440 for the actual number of organisms per gram. The average standard error of log λ is $\sqrt{\cdot 0181} = \cdot 135$, giving limits of $1\cdot 483 \pm \cdot 264 = 1\cdot 219$ and 1·747 for log λ, or 415 and 1400 for the actual number, which may be compared with the more exact limits given by the first method.

Tables IX to XIV. The Probit and Angular Transformations

Many important classes of data may be interpreted on the supposition that a normal deviate is linearly dependent on some observable concomitant measurement, and that an observable frequency is that with which this deviate is exceeded in a normal distribution. For example, the frequency with which a weight is judged to be heavier than a standard increases with the actual weight used; the frequency with which a high jumper clears a rod decreases with the height at which it is placed; the frequency with which insects survive fumigation decreases with the concentration of the fumigant. In many such cases the relation between the frequency and measurement concerned is excellently represented by considering the normal deviate corresponding to the frequency to be a linear function of the measurement, or, more frequently in toxicology, of its logarithm, the measurement being the concentration of the toxin.

Table IX gives the deviate corresponding with each thousandth of the total frequency, which is exceeded by any given frequency; by means of the difference column it is simply used to ten-thousandths. To facilitate standard methods of calculation by making all values positive, 5 has been added to the deviate. The resulting values are known as " probits." It is obvious that the probits corresponding to complementary percentages, such as 40 per cent. and 60 per cent., must add to make 10. Beyond 95 per cent. the actual differences are inserted, and beyond 98 per cent. the tabulation is by ten-thousandths. By subtracting from 10, these advantages may be made available also at the beginning of the table.

Table X, simple quantiles of the normal distribution, gives the deviates corresponding with proper fractions. The proper fractions, up to $\frac{1}{2}$, having denominators up to 30, are here arranged in order of magnitude. When small numbers are involved this table enables the probits to be found more expeditiously than does Table IX.

Example 6

What deviate is exceeded by 4/17 of a normal curve? The deviate given is 0·7215, so that if 4 animals survive out of 17 the probit is 5·7215. The same value may be obtained by expressing 13/17 as a decimal, ·7647, and referring to Table IX.

Dosage Mortality Tests

When tests have been made on several concentrations it is usual to fit a straight line expressing the probit in terms of the logarithmic concentration. The weight assigned to each point will be the number of animals used in determining it multiplied by the weighting coefficient given in Table XI. It is best to draw a provisional line graphically, and obtain the weighting coefficients from the probits read from the graph. Some

workers prefer to draw a provisional line by calculation, using weighting coefficients based on the probits empirically observed. This leads to some ambiguity when at any concentration all die (or all survive), for such observations nominally give an infinite probit with zero weight.

However the provisional line be obtained, for refined purposes, such as tests of significance, it should be replaced by one based on working probits, obtainable from Table XI. When all animals survive, the minimum working probit in the 2nd column corresponding to the expected probit in the 1st is used; when all die, the maximum working probit in the 4th column is used. Other cases are assigned values greater than the minimum working probit by the product of the proportion killed and the range (given in the 3rd column); the same quantity is obtained by decreasing the maximum working probit by the product of the proportion surviving and the range. The working probit is given a weight found by multiplying the total number of animals on which it is based by the weighting coefficient in the 5th column of the table. Any tolerably good provisional line will give a satisfactory result when this process is applied. An excellent account of the treatment of toxicological data has been given by C. I. Bliss (4-7), and the subject is also discussed, from the point of view of amount of information, in *The Design of Experiments* (2).

Example 7

The following survival fractions were obtained on testing sets of eight brine-shrimps, *Artemia salina*, from a single brood, in arsenical solutions having concentrations in geometric progression.

Solution	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Total tested	8	8	8	8	8	8	8
Survived	8	8	6	5	5	1	0

Since, using strengths *C*, *D* and *I*, all or none have survived, the tests with solutions *E* to *H* are available for plotting probit values, in order to obtain a provisional line. A sufficiently good line may usually be drawn graphically. A second serviceable method is to fit a straight line, without weighting, to the probits available. Thus, using Table X, we have

Solution	Empirical Probit.
E	4·33
F	4·68
G	4·68
H	6·15
Total	19·84
Mean	4·96

The slope from four equally spaced points is

$$b = \frac{3(H-E)+(G-F)}{3^2+1^2} = \cdot 546.$$

It is worth while giving some attention to obtaining a satisfactory preliminary fit, since otherwise the fitting process may have to be repeated. It is also desirable to avoid interpolation by choosing simple values. The inclusion of *C*, *D* and *I* will certainly tend to increase the slope, and may lower the average probit. For *b*, therefore, we take the value ·6, and take the probit midway between *F* and *G* to be 4·9. The provisional expected probits are those shown in the 3rd column of Table 2.

TABLE 2.

Solution.	Independent Variate *x*.	Provisional Probit *Y*.	Working Probit *y*.	Weight *w*.	*wy*.
C	3	2·8	2·4081	·7343	1·7683
D	4	3·4	2·9060	1·9002	5·5220
E	5	4·0	4·3775	3·5090	15·3606
F	6	4·6	4·6826	4·8042	22·4961
G	7	5·2	4·6776	5·0193	23·4783
H	8	5·8	6·0998	4·0208	24·5261
I	9	6·4	6·9394	2·4159	16·7649
				22·4037	109·9163

The remainder of Table 2 is then filled by using Table XI.

When all die, as for solution *I*, the working probit has its maximal value, given in the 2nd column of Table XI. This is 6·9394, or ·5394 above expectation. (In the calculations for fitting, we may use either the working probits themselves to obtain the actual constants of the fitted line, or their deviations from provisional expectations to obtain adjustments of the constants provisionally adopted. Table 2 is arranged to show the first method.)

More generally, we subtract from the maximal probit a fraction of the range (3rd column of Table XI) equal to the proportion of survivors; so *G* gives

Maximal working probit	6·2759	Range 2·5573
Range × 5/8	1·5983	
Working probit	4·6776	

while for *H* we have, similarly, 6·0998.

For *E*, if all had *lived*, we should have 3·3443, but as one-quarter have died we add one-quarter of 4·1327, giving 4·3775, while for *F* we find, similarly, 4·6826.

In all cases the weight assigned to the evidence of each solution is the weighting coefficient, corresponding with the provisional probit, multiplied by the number of individuals tested.

From the sum of the weights, and the weighted sums of x and y, we obtain the values of $\bar{x}$ and $\bar{y}$ according to the formulæ:

$$W = S(w), \qquad W\bar{x} = S(wx), \qquad W\bar{y} = S(wy).$$

It is convenient to calculate $S(wx^2)$ at the same time as $S(wx)$; for calculating these any suitable origin such as $F\,(x = 6)$ may be used. The reduced sum of squares of the independent variate is then

$$A = Sw(x-\bar{x})^2 = S(wx^2) - \bar{x}S(wx).$$

With most types of machine, the values of wy will be written down and summed as in Table 2; from these it is easy to obtain the reduced sum of products,

$$B = Swy(x-\bar{x}) = S(wxy) - \bar{x}S(wy).$$

These quantities are all that are needed for adjusting the regression line. For testing goodness of fit we also calculate

$$C = Sw(y-\bar{y})^2 = S(wy^2) - \bar{y}S(wy).$$

Turning to the numerical values of Table 2, we may note that the three solutions first omitted contribute nearly one quarter of the total weight, and a still higher proportion to *A*, or to the amount of information supplied as to the slope of the line. For the others the working probits differ not greatly, but appreciably, from the empirical probits first used.

Numerically, Table 2 gives

$S(wx)$	10·7963	$S(wx^2)$	60·5641	$S(wxy)$	91·1157
$\bar{x}$	·481898	A	55·3614	B	38·1473
$\bar{y}$	4·906167			b	·68906

The recalculated slope, $b = B/A$, is ·68906. This is considerably higher than the provisional value, ·6. Accepting this solution, the 50 per cent. death point is found when $Y = 5$, that is ·093833 above $\bar{y}$. We have, therefore,

$$6{\cdot}481898 + \frac{{\cdot}093833}{{\cdot}68906} = 6{\cdot}618$$

as the estimated 50 per cent. death point.

For goodness of fit we find

$S(wy^2)$	568·6512	
C	29·3835	
$\chi^2 = C - Bb$	3·0977	$n = 5$

Since two constants have been fitted there are five degrees of freedom, so that there is no indication of departure from the form of the theoretical expectation. In these circumstances the variance of the estimated 50 per cent. death point may be taken to be

$$s^2 = \frac{1}{b^2}\left\{\frac{1}{W} + \frac{(x-\bar{x})^2}{A}\right\}$$

Numerically,

$1/W$	·044635
$(\cdot 136)^2/A$	·000334
	·044969
bs	·21206

The standard sampling error of the estimate is therefore ·308.

This is a large-sample method, appropriate if $\bar{y}$ is sufficiently near to 5, and the slope b sufficiently well determined. Failing this, the use of a standard error is not justified, and we should calculate the fiducial limits of the 50 per cent. death point, at any chosen level of significance, as Bliss has proposed (5). These are

$$\bar{x}+\frac{1}{Bb-t^2}\left\{(5-\bar{y})B\pm t\sqrt{A\left\{\frac{1}{W}(Bb-t^2)+(5-\bar{y})^2\right\}}\right\},$$

where t stands for the normal deviate corresponding with the level of significance chosen. *E.g.* the limits are 5·988 and 7·294, when $t = 1{\cdot}96$.

Had χ^2 been excessive, indicating causes of disturbance other than random sampling, the estimated variance would be multiplied by χ^2/n, or, in the small-sample treatment, t would be taken from Student's distribution for the number of degrees of freedom in χ^2, and multiplied by $\chi/\sqrt{n}$ before use in the formula for fiducial limits. It should be remembered also that a significant value of χ^2 may throw some doubt on the linearity of the regression.

The example has been chosen to illustrate the expeditious use of the tables. No interpolation has been employed, as it would be if the process were repeated, using the results of the first fit as a new provisional line. It is, therefore, of interest, as an indication of the precision to be expected, that a much more precise fit gives 6·609±·317 for the 50 per cent. point, showing that this value and its precision have been very satisfactorily estimated. The slope, estimated at ·6891±·1344, is not so good, since the more exact value is ·7126±·1574. Here the error exceeds a tenth of the standard error of random sampling, so that, if the slope were of importance, a second fitting would have been justified. Finally, from the more exact fit we find $\chi^2 = 3{\cdot}773$ in place of 3·098, confirming the agreement with expectation, but showing also that had this been doubtful a more exact fitting might have resolved the doubt.

Dosage Mortality Tests involving Deaths among Controls

A further complication arises when control (untreated) batches of animals give evidence of the occurrence of natural mortality. If the natural mortality rate is M and a proportion p is observed to be killed at any concentration, then the estimated proportion p' that would have been killed by the poison if there were no natural mortality is $(p-M)/(1-M)$; the probit regression line should then be fitted to the probits of p', not to those of p, using modified weighting coefficients obtained by multiplying the coefficients of Table XI by a factor $\dfrac{P}{P+M/(1-M)}$. Values of these reduced weighting coefficients (adapted from Finney (37)) are given in Table XI1 for values of M from ·00 to ·40.

An estimate of the value of M will be provided by the mortality rate in the control group, but this estimate may be modified if, for example, the mortality at low concentrations of the poison is less than that of the control group.

In exact work, and particularly in cases in which M is not well determined by the control group, a more accurate estimate of M may be obtained at the same time as revised estimates of the coefficients a and b of the probit line $y = a+b(x-\bar{x})$ by the method of maximum likelihood. If the number in the control group is n_c, and M_c is the mortality rate in this group, then, taking M_o and $M_o+\delta$ as the first and improved estimates of M, the values of δ, a, and b are given by the three simultaneous equations:

$$\begin{aligned}\left(D+\frac{n_c}{M_o(1-M_o)}\right)\delta+Ea+Fb &= G+\frac{n_c(M_c-M_o)}{M_o(1-M_o)}\\ E\delta+Wa &= W\bar{y}\\ F\delta+Ab &= B\end{aligned}$$

where

$$D=\frac{1}{(1-M_o)^2}\,S\left(w\frac{Q^2}{Z^2}\right),\qquad E=\frac{1}{1-M_o}\,S\left(w\frac{Q}{Z}\right)$$

$$F=\frac{1}{1-M_o}\,S\left(w(x-\bar{x})\frac{Q}{Z}\right),\qquad G=\frac{1}{1-M_o}\,S\left(wy\frac{Q}{Z}\right)$$

Example

The first three columns of Table 2.0 give the results of a spraying test of an extract of a variety of *Derris elliptica* on the saw-toothed grain beetle *Oryzaephilus surinamensis* (Martin (43)). The last line represents the control group, which received the spray medium only; in this group 16·3 per cent. of the insects were affected.

TABLE 2.0

Log. Conc.	No. Exposed	No. Affected	p	p' (M_o = ·163)	Emp. Probit	Y	y	w	wx	wy	Q/Z	$w\,Q/Z$
2·79	125	125	1·000	1·000	—	7·3	7·68	7·9	22·041	60·672	·379	2·994
2·66	117	115	·983	·980	7·05	7·0	7·05	12·8	34·048	90·240	·421	5·389
2·49	127	114	·898	·878	6·16	6·5	6·07	28·2	70·218	171·174	·516	14·551
2·17	51	40	·784	·742	5·65	5·6	5·65	22·4	48·608	126·560	·823	18·435
1·57	132	34	·258	·114	3·79	3·8	3·79	18·3	28·731	69·357	4·557	83·393
—	129	21	·163				W=89·6		203·646	518·003		124·762

$\bar{x} = 2{\cdot}2728$ $\quad E = 124{\cdot}762/{\cdot}837 = 149{\cdot}059$ $\quad n_c/M_o(1-M_o) = 945{\cdot}533$

$S(wx^2)$	$S(wxy)$	$S(wy^2)$	$S(wx\,Q/Z)$	$S(wy\,Q/Z)$	$S(w\,Q^2/Z^2)$
477·492	1219·062	3119·106	229·852	·837)569·530	·700569)406·106
462·854	1177·335	2994·722	283·563		
A = 14·638	B = 41·727	C = 124·384	·837)−53·711	G = 680·442	D = 579·680
			F = −64·171		945·533
					1525·213

Taking M_o as 0·163, the values of p' are calculated from the proportions affected, p. From the empirical probits corresponding to the values of p', a provisional line is drawn and provisional probits Y obtained. If M_o were well determined a second approximation to a and b could then be obtained by the procedure of the previous example, with the exception that the weighting coefficients are taken from Table XI1, interpolating for $M = 0{\cdot}163$. The values of w, wx, and wy, and the sums of squares and products $S(wx^2)$, $S(wxy)$ and $S(wy^2)$ from which A, B and C are derived are shown in Table 2.0.

In order to obtain a second approximation to M, as well as to a and b, the additional columns Q/Z and wQ/Z are required; the values of Q/Z are taken from Table XI1. From these columns the further sums of products $S(wxQ/Z)$, $S(wyQ/Z)$ and $S(wQ^2/Z^2)$ are calculated, thus deriving the values of D, E, F and G as shown.

The equations for δ, a and b are therefore

$$\begin{aligned} 1522{\cdot}3\delta + 149{\cdot}1a - 64{\cdot}17b &= 680{\cdot}44 \\ 149{\cdot}1\delta + 89{\cdot}6a &= 518{\cdot}00 \\ -64{\cdot}17\delta + 14{\cdot}638b &= 41{\cdot}73 \end{aligned}$$

These are best solved by the method outlined in *Statistical Methods*, as the inverse matrix of coefficients, c_{rs}, thus obtained, provides the set of variances and covariances of a and b. The inverse matrix is found to be

$$V = \begin{bmatrix} \cdot0010042 & -\cdot0016711 & \cdot0044024 \\ -\cdot0016711 & \cdot0139415 & -\cdot0073258 \\ \cdot0044024 & -\cdot0073258 & \cdot0876144 \end{bmatrix}$$

giving

$\delta = {\cdot}0014$ $\quad a = 5{\cdot}7789$ $\quad b = 2{\cdot}8570$

The value of M given by the controls is therefore little changed in this example by taking into account all the information provided by the data.

The data can be tested for goodness of fit by calculating the expected numbers of affected and non-affected insects in each group, and deriving a value of χ^2 from the differences between observed and expected numbers in the ordinary manner. Classes with very small expectation should be grouped before calculating χ^2. In the present example the value of χ^2 so found (the heaviest concentrations being grouped) is 4·095. This is associated with 2 degrees of freedom. There is thus no evidence of heterogeneity.

The 50 per cent. affected point is given by

$$x-\bar{x} = (5-a)/b.$$

Hence $x = 2{\cdot}000$. The variance of this estimate is (on large sample theory)

$$s^2 = \frac{1}{b^2}\{c_{22}+2c_{23}(x-\bar{x})+c_{33}(x-\bar{x})^2\}$$

$$= \frac{1}{(2{\cdot}8570)^2}\{{\cdot}01394{\cdot}15+2\times{\cdot}272629\times{\cdot}0073258+({\cdot}272629)^2 \times{\cdot}0876144\}$$

$$= {\cdot}002995$$

the value of the c's being taken from the matrix V above. The variance of the improved estimate of natural mortality is $c_{11} = {\cdot}001004$, corresponding to a standard error of ·0317.

Other Transformations

The normal transformation may be looked upon as one of a series of methods of transforming percentages into more suitable variables. Of these the transformation of a probability varying from 0 to 1 into an angle varying from 0° to 90°, according to the relationship

$$p = \sin^2\phi$$

is remarkable in that the amount of information concerning ϕ is constant. In consequence, in large samples, the sampling variation tends to normality with a variance which depends only on the number of observations on which the percentage is based. The weighting coefficient for this transformation is constant. In other respects Tables XII to XIV afford the same facilities for handling percentages transformed in this way as do Tables IX to XI for the normal transformation. This transformation is equivalent to replacing the normal curve by one of which the ordinate is $\sin 2\phi$.

A third transformation of some interest which is readily available from these tables, is given by

$$z = \tfrac{1}{2}(\log_e p-\log_e q)$$

or

$$p = e^{2z}/(1+e^{2z}).$$

If $r = 2p-1$, the values of z are given in Table VII. With this transformation we may use

Maximal working z	$Z+\frac{1}{2P} = Z+\frac{1}{1+R}$
Range	$\frac{1}{2PQ} = \frac{2}{1-R^2}$
Weighting coefficient	$4PQ = 1-R^2$

where Z is the expected value of z given by any provisional fitting, and the value of R is taken from that of Z, using Table VII. This transformation is equivalent to replacing the normal curve by one of which the ordinate is $\frac{1}{2}\,\mathrm{sech}^2 z$.

In addition to their toxicological and similar applications, transformations are of use in the analysis of material which has a very skew distribution, for the purpose of equalising the variance of quantities of widely different magnitudes which it is desired to include in the same analysis, and for the purpose of reducing the observations to a scale on which the effects of associated variables are more nearly constant. Thus when it is required to perform an analysis of variance on a set of percentages, *e.g.* of the deaths of seedlings obtained from plots of a replicated experiment, it is frequently advantageous to transform these percentages by means of the angular transformation. As we have seen, this has the effect, in large samples, of equalising the component of variance due to sampling, when each percentage is based on the same number of observations. The square root transformation

$$y = \sqrt{x}$$

has the same property when the observations on each plot are distributed according to a Poisson distribution. This transformation is equivalent to the angular transformation at each end of the percentage scale. (In this connection see W. G. Cochran (8).)

The logarithmic transformation

$$y = \log x$$

is useful when dealing with material on which the effects of other variables may be expected to be proportional to the numbers involved. The influence of a given increase in temperature on the number of insects caught in a light trap, for example, is likely to be proportional to the number of active insects, in which case the increment in the logarithm of the catch will be the same whatever the number of insects, unless some of the numbers are very small. In this connection it may be noted that the transformation $y = \log (x+1)$ is sometimes convenient, being substantially equivalent to $y = \frac{1}{3}\sqrt{x}$ when x is less than 10 and to $y = \log x$ for larger numbers.

A further use of transformations is for the purpose of facilitating interpolation by obtaining a transformed function which is much more nearly linear. One example of this, the use of the reciprocals of n_1 and n_2 in interpolation in the table of z (Table V), has already been mentioned. The logarithmic transformation and the transformation of Table VII have frequently proved of use when dealing with functions whose argument is a probability.

Table XIV1. Scores for Linkage Data from Intercrosses

Much linkage data with plants is obtained by selfing double heterozygotes, which may be in coupling or in repulsion. In some cases, especially with animals, single or double backcrosses may also be available. The compilation of such complex data, for the estimation of linkage, or the examination of heterogeneity, is much facilitated by efficient scores, based on the method of maximum likelihood, such as have been developed for human genetics.

Such scores possess the properties (i) that the scores are linear functions of the frequencies observed, (ii) that the expected score is zero if the trial value is correct, (iii) that the ratio of change of the expected score with change of trial value is equal to its sampling variance, and to the amount of information to be expected from such a body of data as is available. Estimation from a single body of data thus consists in finding the value for which the total score vanishes. The efficiently weighted combination of different lots of data consists simply in adding their scores. Homogeneity is tested by recognising that the square of the score divided by the amount of information is χ^2 for one degree of freedom. If m is the expectation of any distinguishable class, expressed as a function of the unknown parameter θ, the efficient score for cases observed in that class may be taken to be

$$\frac{1}{m}\frac{\partial m}{\partial \theta}.$$

For double heterozygotes intercrossed the two singly recessive classes have the same expectation and the same score. Table XIV1 gives the means of scoring such data in coupling and repulsion for each 1 per cent. of the recombination fraction, y, the quantities tabulated being :

	Coupling	Repulsion
Double dominant	$\frac{2(1-y)}{2+(1-y)^2}$	$-\frac{2y}{2+y^2}$
Single recessives	$\frac{1}{y}-\frac{1}{2-y}$	$\frac{1}{1-y}-\frac{1}{1+y}$
Double recessive	$\frac{2}{1-y}$	$-\frac{2}{y}$

The signs in the table are for $\theta = 1-y$, where y is the recombination fraction, in order that evidence for linkage ($y<\cdot 5$) may be supplied by positive scores at 50 per cent.

Data should be scored at two adjacent values giving positive and negative scores. One hundred times the difference then gives the amount of information to a good approximation. For more accurate work, and especially for close linkage, four adjacent values with four-point interpolation may be used. The four-point value for the actual amount of information obtained at the value $x+\theta$ is

$$(u_{x+1}-u_x)-\frac{1}{6}(1-3\theta^2)\delta^2 u_{x+1}+\frac{1}{6}(1-3\phi^2)\delta^2 u_x,$$

where $\theta+\phi = 1$.

Example :

For two factors in the Sweet Pea we have the data :

	Double dominant	Single recessives	Double recessive
Coupling	268	30	79
Repulsion	324	344	0

Scoring these at 6 per cent., 7 per cent., 8 per cent. and 9 per cent., we find

	6 per cent.	7 per cent.	8 per cent.	9 per cent.
Score	−119·701	−43·367	15·165	61·852
δ^1	+76·334	+58·532	+46·687	
δ^2		−17·802	−11·845	

To find the score at an intermediate value, such as 7·811 per cent., we find the inner interpolate, 4·102, and, since the linkage is close, the four-point value, 5·191. A method of calculating this is given on p. 24 of second edition (at present altered to p. 26) section on *Interpolation*. The first approximation to the amount of information obtained is 5853·2, the adjustment is −457·0, giving 5396·2 at this value.

Tables XV and XVI. Latin Squares

Latin squares of size 7×7 and smaller have been enumerated (11), (22). For 4×4 and 5×5 squares all the standard squares (*i.e.* squares with first row and first column in the standard order) are shown in Table XV, except that only one of each pair of conjugate 5×5 squares is given (two standard squares are conjugate if the rows of one are the columns of the other). From each 4×4 standard square 4 ! 3 ! −1 non-standard squares, all different, may be generated by permuting all the rows except the first, and all columns. Thus each 4×4 standard square represents a set of 144 squares. There are consequently 4×144 or 576 4×4 Latin squares. Similarly, there are 56×2880 or 161,280 5×5 Latin squares. There is one 2×2 standard square and one 3×3 standard square, giving two 2×2 squares and twelve 3×3 squares in all.

For 6×6 squares one square of each transformation set (*i.e.* set generated from one of its members by permutation of rows, columns, and letters), or of each pair of conjugate transformation sets, is shown. Such permutation of rows, columns and letters will generate $(6\,!)^3$ or 373,248,000 squares, of which 6×6 ! or 4320 are standard squares. These squares are, however, not necessarily all different, the largest number actually obtained in any one 6×6 set being one-quarter of the possible number, *i.e.* 1080 standard squares, and the smallest number one 216th, *i.e.* 20 standard squares. The actual numbers in each transformation set are shown in the table. There are in all 9408 standard 6×6 squares, giving 812,851,200 6×6 squares in all.

By permuting with each other the three categories, rows, columns, and letters, six sets are formed, not necessarily all different. These sets are said to be *adjugate*. In the case of the 6×6 squares each pair of conjugate sets and the set immediately following them in the table form an adjugate trio of sets. The remaining sets are self-adjugate.

Norton (1939) has enumerated 562 sets of 7×7 squares, of 146 species, containing 16,927,968 standard squares. Four examples are shown in the table. Square (*a*) is a member of a transformation set in which the squares generated by the permutation of rows, columns and letters are all different. It thus contains $(7\,!)^3$ or 128,024,064,000 squares, of which 7×7 ! or 35,280 are standard squares. The sets generated by the adjugate permutations are also all different. The example thus stands for an aggregate of 211,680 standard squares. No Graeco-Latin squares can be formed with any of these.

Square (*b*) differs from square (*a*) only by the interchange of two pairs of letters, *B* and *C* in the 5th and 6th columns. It belongs to a set of only 11,760 standard squares, which is one of an adjugate trio of sets, giving 35,280 standard squares in all. One-seventh (1680) of the standard squares of one of these sets are self-conjugate. Again no Graeco-Latin square can be formed.

The third illustration is a Graeco-Latin square, with the fourth category (Greek letters) represented by numeral suffices. No fifth category can be added. The suffices are standardised by assigning them in their natural order to the first row. The Latin square involved is one of a set of only 210 standard Latin squares, but the same transformation of the Graeco-Latin square yields a set of 1680 standard Graeco-Latin squares, containing eight different standard arrangements of the suffices in each Latin square. These eight solutions are in fact all that exist, so that this set of Latin squares belongs to only one Graeco-Latin set. Permutation of the three categories, rows, columns, and letters, yields the same set, but if suffices are interchanged with rows, columns, or letters, three more sets of 1680 standard Graeco-Latin squares are formed (6720 in all).

The three new sets all involve the cyclic Latin square shown as the fourth example. There are only 120 standard Latin squares in this set, so that each has 14 solutions in each of the Graeco-Latin sets, or 42 in

all. Under orthogonal squares are given six 7×7 squares, of which the first is the above Latin square, while the rest are five more Graeco solutions which belong to a second set and are mutually orthogonal.

Little is known about squares larger than 7×7, other than the orthogonal properties of the special sets shown in Table XVI. In Table XV a single example of each size from 8×8 to 12×12 is given. These examples will suffice (after permutation of rows, columns, and letters at random) for experimental arrangements requiring Latin squares of these sizes.

To select a square at random from all possible squares of a given size up to 6×6 proceed as follows :

(1) Select one of the given squares by using the key numbers printed below each square, selecting a number at random from all possible key numbers. If the key number falls in the second of two groups use the conjugate square.

(2) In the case of 3×3, 4×4, and 5×5 squares permute all rows except the first of the selected square, and all columns. Alternatively permute all rows except the first and assign the letters to the treatments at random. For 6×6 squares permute all rows and all columns at random, and then assign the letters to the treatments at random.

Table XVI gives complete sets of orthogonal Latin squares of sides 3, 4, 5, 7, 8, and 9. Two Latin squares are orthogonal to each other if, when they are superimposed, every letter of one square occurs once and once only with every letter of the other. Such a pair of squares (one square being written with Greek and the other with Latin letters) form a Græco-Latin square.

If a complete set of $p-1$ orthogonal squares of side p exists, the p^2-1 degrees of freedom corresponding to the contrasts of the p^2 cells of the square split up into $p+1$ sets of $p-1$ degrees of freedom, corresponding to the contrasts between the rows, between the columns, and between the letters of each of the $p-1$ squares. A set of less than $p-1$ orthogonal squares gives a similar but incomplete partition.

Complete sets of orthogonal squares are known to exist for all prime numbers and for $p = 4$, 8 and 9. (See *The Design of Experiments*.) The rule of formation for all prime numbers will be apparent from the inspection of the sets for $p = 5$ and $p = 7$, each row being formed by moving the previous row 1, 2, 3 . . . places to the left.

The rows of the 8×8 squares may be taken to represent the 8 combinations of three two-level factors A, B, and C, in the order (1), a, b, ab, c, ac, bc, abc, and the columns may be taken to represent the similar combinations of a further three factors D, E, and F. The 64 cells of each square will then represent the 64 combinations of the 6 factors. The numbers of each square will divide these combinations into 8 groups of 8, the contrasts among which will represent a set of 7 degrees of freedom orthogonal to A, B and C and their interactions, and to D, E, and F and their interactions, and also to the six other sets of seven degrees of freedom given by the other six squares. In the squares shown these degrees of freedom correspond to specific interactions between the six factors. The interactions corresponding to 3 degrees of freedom from each set are shown in the table, the remainder being given by the " generalised " interactions of these three. Note the law of formation of each set from the preceding one, corresponding to the cyclic interchange of all the rows of the square except the first.

The set of 9×9 squares possesses similar properties. The actual degrees of freedom confounded, when the cells of the squares represent the 81 combinations of four factors at three levels, are given in terms of the notation developed by Yates (3).

Besides their use in ordinary factorial design, complete sets of orthogonal squares serve as the basis of quasi-factorial and quasi-Latin square designs (balanced lattices and lattice squares) and also serve to generate other designs in balanced incomplete blocks. (See (14), (15), (16), (17), and Tables XVII-XIX.) Pairs and larger incomplete sets of orthogonal squares (Græco- and hyper-Græco-Latin squares) also provide designs for varietal trials superseding the so-called semi-Latin squares (3).

An orthogonal pair of squares exists for all odd numbers, being formed of two " diagonal " squares (squares of the type of the 5×5 squares I and IV). Such a pair also exists for all numbers which are multiples of 4. The process of generation, which depends on the existence of orthogonal pairs of squares for factors of the number, may be illustrated for a pair of 12×12 squares. Let α and α' represent the 4×4 squares I and II, β and β', and γ and γ', the same squares with 1, 2, 3, 4 replaced by 5, 6, 7, 8 and 9, 10, 11, 12 respectively. Then

$$\begin{array}{cccc} \alpha & \beta & \gamma & \\ \beta & \gamma & \alpha & \text{and} \\ \gamma & \alpha & \beta & \end{array} \qquad \begin{array}{ccc} \alpha' & \beta' & \gamma' \\ \gamma' & \alpha' & \beta' \\ \beta' & \gamma' & \alpha' \end{array}$$

form an orthogonal pair of 12×12 squares. The process will generate larger orthogonal sets if such sets exist for both factors of the number. Thus there will be a set of three squares of side 20.

It has been shown that no orthogonal pair of 6×6 squares exists, but larger even numbers not divisible by 4 have not been investigated.

Tables XVII to XIX. Balanced Incomplete Blocks

If v varieties (or treatments) are to be compared in randomized blocks of k experimental units (k being less than v), block differences can be simply eliminated if the arrangement is such that every two varieties occur together in the same number (λ) of blocks. If r is the number of replicates, and b the number of blocks, the number of experimental units is $rv(=bk)$, and $\lambda = r(k-1)/(v-1)$. Examples of experimental material and processes in which this type of arrangement, called *balanced incomplete blocks*, is likely to be of use are: litters of animals, monozygotic twins and other paired material, laboratory and technological processes in which the nature of the process or the apparatus available imposes a definite limit on the number of treatments in a group, agricultural experiments on commercial farms when each farm is only prepared to undertake a small number of treatments, and varietal trials involving a large number of varieties. (See also Yates (16).)

Arrangements satisfying the required conditions are clearly provided by taking all possible combinations of the v varieties, k at a time, but if v is at all large, the number of replicates required will be very large. Arrangements involving fewer replicates are therefore of importance. Tables XVIII and XIX give indexes, by number of replications (r), and by number of units in a block (k) respectively, of the arrangements requiring ten or less replications which are known to exist, and of all arithmetically possible arrangements, the existence of which had not been disproved at the date of the first edition. Since then the non-existence of solutions of Nos. 10 and 14 has been demonstrated by Q. M. Hussain (44, 45), and No. 8 has been eliminated by H. K. Nandi (46). These cases are marked by an asterisk. Cases not yet solved are marked by a dash. General discussions of the construction of incomplete block designs have been given by R. C. Bose (31), and C. R. Rao (47). The solutions for Nos. 26 and 27 are due to Rao, those for Nos. 17 and 20 to K. N. Bhattacharya (32).

The combinatorial solutions which are referred to by numbers are shown in Table XVII. Those involving all combinations are indicated by *u* (unreduced), and the two series derivable from sets of orthogonal squares (described below) by *o.s.* When $k>\frac{1}{2}v$ arrangements can be obtained from arrangements for the same number of varieties in blocks of $v-k$ units, by replacing each block by its complement, *i.e.* by a block containing all the varieties missing from the original block.

Some of the combinatorial solutions are derivable very simply by cyclic substitutions. In such cases one block only for each family is given in Table XVII. In solution number 2, for example, the block *aefgi* is given. The contents of the other 10 blocks are obtained progressively by replacing each letter by the next in order, with the convention that the last letter k is followed by a. Thus the blocks are *aefgi*, *bfghj*, *cghik*, *dhija*, etc.

For the dicyclic solutions a similar procedure holds. In solution number 5, for example, the 16 varieties are represented by 4 letters in combination with 4 suffixes. The 16 blocks are generated by performing the cyclic substitution on the letters (thus obtaining 4 blocks) and then performing the cyclic substitution on the suffixes of each of these 4 blocks.

Some of the cyclic and dicyclic solutions consist of more than one family, each family being generated by a cyclic or dicyclic substitution of the above type.

Two series of arrangements and their complements are derivable from complete sets of orthogonal Latin squares (Table XVI). In the first, for p^2 varieties in $p(p+1)$ blocks of p plots ($r=p+1$, $\lambda=1$), each of the p^2 cells of the set of squares is taken to represent a variety, the blocks being determined by the rows, the columns and the numbers of the $p-1$ squares. In the second, for p^2+p+1 varieties in p^2+p+1 blocks of $p+1$ plots ($r=p+1$, $\lambda=1$), p^2 of the varieties are divided as in the first arrangement, each of the $p+1$ additional varieties being allotted to one of the $p+1$ sets of blocks so formed, while the last block consists entirely of these additional varieties. Thus the set of 4×4 squares gives the following arrangements for 16 and 21 varieties (the latter including the bracketed varieties):

a b c d (*q*)	*a e i m* (*r*)	*a f k p* (*s*)	*a g l n* (*t*)	*a h j o* (*u*)	(*q r s t u*)
e f g h (*q*)	*b f j n* (*r*)	*b e l o* (*s*)	*b h k m* (*t*)	*b g i p* (*u*)	
i j k l (*q*)	*c g k o* (*r*)	*c h i n* (*s*)	*c e j p* (*t*)	*c f l m* (*u*)	
m n o p (*q*)	*d h l p* (*r*)	*d g j m* (*s*)	*d f i o* (*t*)	*d e k n* (*u*)	

The second of these series, at least up to 10 replications, may also be generated by the cyclic substitutions shown in Table 2.1. For convenience the varieties are here represented by numbers instead of letters. The first series may then be obtained from the second by deleting any set of varieties occurring in the same block.

TABLE 2.1. CYCLIC SOLUTIONS FOR BALANCED INCOMPLETE BLOCKS

Problem.	Modulus (v).	Selected numbers.
O. S. 2	7	1, 2, 4
O. S. 3	13	1, 2, 4, 10
2	11	1, 5, 6, 7, 9
O. S. 4	21	1, 4, 5, 10, 12
O. S. 5	31	1, 2, 4, 11, 15, 27
9	15	1, 2, 3, 5, 6, 9, 11
O. S. 7	57	1, 4, 6, 14, 15, 21, 33, 37
19	19	1, 3, 5, 6, 7, 8, 11, 14, 15
23	37	1, 2, 4, 8, 18, 25, 26, 30, 36
O. S. 8	73	1, 2, 4, 8, 16, 32, 37, 55, 64
O. S. 9	91	1, 2, 7, 11, 24, 27, 35, 42, 54, 56

See also the dicyclic solutions in one family (No. 5), in two families (No. 13), and the cyclic solution in two families (No. 29) of Table XVII.

Youden's Squares

It has been shown (Youden, 21) that when $b = v$ it is possible, at least in the cases of practical importance, so to arrange the order of the varieties in the blocks that each variety occurs once in each position in the block, and it is thus possible to eliminate experimental differences between these positions, each of which comprises a complete replication.

For example, experimental material consisting of 13 plants each with 4 corresponding leaves may be assigned to 13 treatments so that each treatment occurs once on each type of leaf. At the same time the 13 plants constitute randomized blocks of 4 treatments each, in which each treatment occurs on the same plant once with every other treatment. Similarly a varietal trial of 13 varieties in blocks of 4 may be arranged on the ground in a 4×13 rectangle of plots so that both row and column differences are eliminated from the varietal comparisons.

In drawing up an actual experimental arrangement of this type the blocks should be assigned to the plants (or arranged on the ground) at random, and the experimental treatments (or varieties) should be similarly assigned to the letters.

A cyclic or dicyclic substitution which gives an incomplete block solution will also give the corresponding Youden square, provided that each block is written in the order in which it is generated. The complementary Youden square solution may be obtained by performing the same cyclic substitution on the block which is complementary to the given block.

Cyclic solutions in two families provide an extension of Youden squares. In these the differences corresponding to the positions in the blocks of each family may be eliminated, *e.g.* 9 degrees of freedom in solution No. 29 for 10 replicates of 41 varieties.

Besides balanced incomplete blocks and Youden squares a number of different types of non-balanced arrangement are also available, namely quasi-factorial arrangements, described in (25) and (26), lattice squares (27), and partially balanced arrangements (28). The various types of arrangement are reviewed in (3) and (29). The experimenter should acquaint himself with their advantages and disadvantages before deciding which is most appropriate for his own needs.

Analysis of Balanced Incomplete Block Experiments

In the analysis of incomplete block experiments the block differences may be eliminated entirely from the varietal comparisons (Yates, 16), or, as has been shown later (Yates, 23), the information contributed by these comparisons may be combined with that provided by the intra-block comparisons, with weights depending on the relative accuracy of the two sets of comparisons (as estimated from the experimental results).

If the choice of blocks has been successful in substantially reducing the variability of the material, then the amount of information contributed by the inter-block comparisons will be small, but in the limiting case when the inter-block and intra-block comparisons are of equal accuracy, the fraction of the total information contained in the inter-block comparisons is given by the efficiency factor $E = \dfrac{1-1/k}{1-1/v} = \dfrac{v\lambda}{rk}$, which approaches the value $\frac{1}{2}$ when $k = 2$ and v is large.

In general, therefore, it is worth recovering the inter-block information, except in small experiments with less than 10 degrees of freedom for blocks, in which the relative weights cannot be determined with sufficient accuracy. If the method described below is followed the amount of additional computation is not great.

Sometimes the blocks may be arranged in groups so as to minimise the inter-block variance. Each such group must contain an equal number of replicates of each variety. In general it is not possible so to arrange things that each group of blocks contains only a single replication. Such groups as are possible in the designs of Table XVII are indicated in the table itself. Within each group the arrangement of the blocks should be at random.

If V_s be the sum of all the yields of variety s, T_s the sum of all the block totals of blocks containing variety s, T'_s the sum of all the remaining block totals, and G the total yield of all plots, the estimates of the varietal differences derived from the intra-block comparisons are obtained from the quantities

$$Q_s = V_s - T_s/k,$$

or, as is more convenient when $k > \frac{1}{2}v$,

$$Q'_s = V_s + T'_s/k.$$

The actual differences in units of the total yield of the r replicates are given by the differences of

$$Q_s/E \text{ or } Q'_s/E,$$

the sum of the first set being zero, and the second set $rG/\lambda v$.

The error variance of these latter sets of quantities is r/Ew, where $1/w$ is the intra-block error variance.

The estimates of the varietal differences derived from the inter-block comparisons are similarly given by the differences of

$$rT_s/(r-\lambda) \text{ or } rT''_s/(r-\lambda),$$

in units of the total yield of r replicates. The error variance of these sets of quantities is $kr^2/(r-\lambda)w'$, where $1/w'$ is the error variance of the inter-block comparisons, in units of a single plot.

The most efficient estimates of the varietal differences are given by the quantities

$$Y_s = V_s + \mu W_s$$

where

$$W_s = (v-k)V_s - (v-1)T_s + (k-1)G,$$

and

$$\mu = \frac{w-w'}{wv(k-1)+w'(v-k)}.$$

The error variance of the $Y's$ is

$$\frac{kr(v-1)}{wv(k-1)+w'(v-k)}.$$

The analysis of variance follows the lines shown in the accompanying table, in which dev^2 indicates the sum of squares of the deviations, y the individual yields and B the block totals. If recovery of inter-block information is not required then only the part (a) of the analysis of variance is required. There is also no need to subdivide the sum of squares for blocks, ignoring varieties. The comparison of the mean square (a) for varieties with intra-block error gives an exact test of significance for the intra-block estimates.

Method (a).	D.F.	S.S. (a).	S.S. (b).	Method (b).
Blocks (ignoring varieties) :				Blocks (eliminating varieties) :
Varietal component . .	$v-1$	$\frac{dev^2\ T}{k(r-\lambda)}$	$\frac{dev^2\ W}{rv(v-k)\ (k-1)}$	Varietal component
Remainder	$b-v$	†	→ †	Remainder
Total	$b-1$	$\frac{dev^2\ B^*}{k}$	†	Total
Varieties (eliminating blocks) .	$v-1$	$\frac{dev^2\ kQ}{k^2\ rE}$	$\frac{dev^2\ V}{r}$	Varieties (ignoring blocks)
Intra-block error . . .	$rv-v-b+1$	†	←→ †	Intra-block error
Total	$rv-1$	$dev^2\ y^*$	←→ $dev^2\ y^*$	Total

* Requires checking.

† Calculated by addition or subtraction.

If the mean square for intra-block error is M and the mean square (b) for the $b-1$ degrees of freedom for blocks, eliminating varieties, is M'', w and w' are estimated from the equations

$$w = \frac{1}{M}\ ,\quad w' = \frac{v(r-1)}{k(b-1)M''-(v-k)M}$$

If M'' is less than M, w' may ordinarily be taken as equal to w.

If the structure of the design is such that the blocks fall into r groups, each containing a single replication of all the varieties (as in solutions Nos. 6, 7 and 25), and each of these groups of blocks is itself arranged in a large block, then the sum of squares corresponding to the $r-1$ degrees of freedom for complete replications must be eliminated from the sums of squares for blocks. The formula for w' will then become

$$w' = \frac{r-1}{rM''-M}$$

In the more general case in which there are c such groups of blocks, each containing r/c replications, we have

$$w' = \frac{v(r-1)-k(c-1)}{k(b-c)M''-(v-k)M}$$

In the case of a Youden square the sum of squares corresponding to the $k-1$ degrees of freedom for the k series (representing leaf position, etc.) must be eliminated from the intra-block error.

Example 8

Table 3 shows the scores of 18 litters of 4 rats in a discrimination test, the values shown being the square roots of the number of trials prior to 12 consecutive successful trials (the data are a random selection from scores published by F. A. E. Crew (10)). A dummy trial of 9 treatments (*e.g.* dietary treatments) in blocks of 4, corresponding to litters, is superimposed on these scores. (Design 11 of Table XVII, $r = 8$, $v = 9$, $k = 4$, $b = 18$, $\lambda = 3$.)

TABLE 3. SCORES OF 18 LITTERS OF RATS

f 2·6	*f* 5·9	*a* 7·0	*i* 2·4	*i* 5·0	*d* 10·1	*b* 3·9	*h* 4·0	*b* 2·8
d 9·7	*g* 2·6	*f* 4·6	*d* 4·0	*h* 7·4	*a* 9·7	*d* 4·1	*f* 6·1	*f* 2·6
c 5·4	*i* 5·9	*i* 4·9	*g* 3·0	*e* 10·3	*f* 5·7	*e* 6·4	*g* 4·4	*e* 2·8
e 6·9	*b* 6·3	*c* 3·3	*f* 2·4	*c* 9·4	*h* 7·5	*i* 6·3	*c* 3·3	*h* 3·3
24·6	20·7	19·8	11·8	32·1	33·0	20·7	17·8	11·5
b 5·7	*b* 4·7	*a* 3·0	*c* 7·5	*c* 3·7	*i* 3·0	*d* 4·5	*g* 2·6	*b* 7·3
h 9·3	*g* 6·6	*h* 1·4	*g* 2·2	*a* 5·2	*g* 2·6	*b* 6·0	*e* 4·9	*e* 5·4
c 5·4	*a* 5·5	*i* 4·2	*e* 2·6	*d* 2·4	*a* 4·7	*g* 4·6	*d* 6·0	*f* 5·7
i 6·1	*h* 5·3	*d* 2·8	*a* 4·4	*b* 2·4	*e* 2·4	*c* 3·3	*h* 4·6	*a* 4·4
26·5	22·1	11·4	16·7	13·7	12·7	18·4	18·1	22·8

Grand Total : 354·4. Mean : 4·92.

Table 4 shows the values of V, T, $4Q$ and W for the nine treatments *a-i*. The analysis of variance is shown in Table 5.

TABLE 4. CALCULATION OF ADJUSTED SCORES IN DISCRIMINATION TEST

	V.	T.	$4Q = 4V-T.$	$Q/E+\frac{1}{6}G.$	$W = 5V-8T+3G.$	$Y = V+\mu W.$
a	43·9	152·2	+23·4	46·3	+65·1	45·7
b	39·1	156·4	0	39·4	+7·5	39·3
c	41·3	169·6	−4·4	38·1	−87·1	38·9
d	43·6	151·7	+22·7	46·1	+67·6	45·4
e	41·7	159·2	+7·6	41·6	−1·9	41·6
f	35·6	162·0	−19·6	33·6	−54·8	34·1
g	28·6	138·3	−23·9	32·3	+99·8	31·3
h	42·8	172·5	−1·3	39·0	−102·8	40·0
i	37·8	155·7	−4·5	38·0	+6·6	38·0
	354·4	1417·6	0	354·4	0	354·3
Divisor	8	4·5 = 20	$4^2 \cdot 8 \cdot 27/32$ = 108		8·9·5·3 = 1080	

TABLE 5. ANALYSIS OF VARIANCE, DISCRIMINATION TEST

	D.F.	S.S. (a).	S.S. (b).	M.S.
Blocks :				
Varietal component .	8	41·4684	37·0634	4·6329 (*b*)
Remainder . .	9	138·2011	138·2011	15·3557
Total . . .	17	179·6695	175·2645	10·3097 (*b*)
Varieties . . .	8	19·6044	24·0094	2·4506 (*a*)
Error	46	119·4506	119·4506	2·5968
Total	71	318·7245	318·7245	—

From the results of the analysis of variance we obtain

$$w = \frac{1}{2\cdot5968} = 0\cdot3851, \quad w' = \frac{63}{68\times10\cdot3097-5\times2\cdot5968} = 0\cdot0916$$

$$\mu = \frac{0\cdot3851-0\cdot0916}{27\times0\cdot3851+5\times0\cdot0916} = \frac{0\cdot2935}{10\cdot8557} = 0\cdot02704$$

The final adjusted scores in terms of the total scores of eight rats are given in the last column of Table 4. The standard error of these scores is

$$\sqrt{\frac{256}{10\cdot8557}} = \sqrt{23\cdot58} = 4\cdot86$$

The scores based on intra-block information only (which are equal to $Q/E+\frac{1}{6}G$) are also shown in Table 4. Their standard errors are

$$\sqrt{(8\times2\cdot5968\times27/32)} = \sqrt{24\cdot62} = 4\cdot96$$

Thus the gain in information from the recovery of the inter-block information is 24·62/23·58−1 or 4·4 per cent. (excluding losses due to inaccuracy of weighting). The gain is here trivial, but if inter-litter and intra-litter comparisons had been of equal accuracy, the gain would have been 18·5 per cent.

The litter totals can be adjusted to allow for the effects of the treatments by deducting one-eighth the scores based on intra-litter differences only of each of the four treatments contained in the litter concerned from the total of that litter (adding four times the general mean if desired). Thus the adjusted total of the first litter is 24·6−4·20−5·76−4·76−5·20+4×4·92 = 24·36.

TABLES XX AND XXI. ORDINAL (OR RANKED) DATA

It is often necessary to draw statistical conclusions from data giving the order of a number of magnitudes without knowledge of their quantitative values. Thus in tests of psychological preference subjects can often express preferences, without being able to assign numerical values to the force with which the preference is felt. Not infrequently, also, an experimenter who possesses quantitative values may suspect that the metric

used is unsuitable to the comparisons he wishes to make, and prefer to draw conclusions only from the order of the magnitudes observed.

The analysis of such data is greatly facilitated by Table XX, which gives the average deviate of the rth largest of samples of n observations drawn from a normal distribution having unit variance. Symbolically this comes to

$$\mu_r = \int_{-\infty}^{\infty} \frac{n!}{(r-1)!(n-r)!} p^{n-r} q^{r-1} xz \, dx,$$

where z is the ordinate of the normal curve, and p and q are the probabilities respectively of falling short of and exceeding x.

From the order in which any subject places a series of objects scores may now be assigned. With a number of different subjects, the experimenter is then in a position to test, for example, whether the scores assigned to any one object significantly exceed those assigned to another; or, more generally, to apply an analysis of variance to determine whether the variance in score among objects exceeds the remainder due to differences in the order chosen by different subjects. More elaborate analyses afford tests of differentiation in preference between classes of subjects of different sex or age. The labour of summing the squares of individual scores is saved by using the sums of squares given in Table XXI.

The scoring of ties causes little difficulty. If, for example, the fifth and sixth objects out of 13 are judged to be equal, instead of scoring one ·39 and the other ·19, each is given the same score ·29. In such cases the sum of squares needs adjustment. When only two objects are judged to be equal it is easy to see that the reduction is $\frac{1}{2}(\cdot 39 - \cdot 19)^2 = \cdot 02$, so that the sum of squares is 10·7904 in place of 10·8104 shown by the table.

Table XXII. The Leading Differences of Powers of Natural Numbers

The table gives values of $\Delta^r 0^s / r!$, where $\Delta^r 0^s$ denotes the leading rth difference of the sth powers of the natural numbers commencing at zero. Its application to combinatorial problems arises from the identity

$$\frac{\Delta^r 0^s}{r!} \equiv \sum_{s/r} \frac{s!}{(p_1!)^{\pi_1}\pi_1!(p_2!)^{\pi_2}\pi_2! \ \ldots}$$

where $(p_1^{\pi_1} p_2^{\pi_2} \ldots)$ is a partition of the number s into r parts, *i.e.*

$$\Sigma \pi p = s, \qquad \Sigma \pi = r.$$

The partitional function on the right represents the number of ways of dividing s persons into the r parties specified by the partition. The summation proceeds over all partitions of s into r parts.

Hence $\Delta^r 0^s / r!$ is the number of ways of dividing s persons into r parties of all possible sizes. For example, with $s = 4$, $r = 2$, the table gives 7 ways, which are in fact,

$$(ab)(cd),\ (ac)(bd),\ (ad)(bc),\ (abc)(d),\ (abd)(c),\ (acd)(b),\ (bcd)(a).$$

If s objects are equally likely to fall into any of n classes, the probability that they will fall into exactly r of the possible n classes can be shown to be

$$\frac{1}{n^s} \frac{n!}{(n-r)!} \frac{\Delta^r 0^s}{r!}.$$

Hence the distribution of r can be derived from the table. Thus the table is useful for testing whether objects fall together in classes too often or too rarely. For practical applications see Stevens (12), (13).

Example 9

In the expression for π it may be noticed that in the first 20 decimal places all digits save zero occur at least once. Zero is the only digit absent. It is not obvious how frequently 0, 1, 2, . . . absent digits would be found in a series chosen at random.

Putting $s = 20$, $n = 10$, $r = 9$ in the expression shown we find

$$\frac{10!}{10^{20}} \times 1201 \cdot 13 \times 10^{10}$$

$$= 3 \cdot 6288 \times \cdot 120113 = \cdot 4359.$$

Hence in 43·59 per cent. of trials we should expect exactly one absent digit; similarly we should expect no digit absent in 21·47 per cent. of such trials; consequently two or more digits will be absent in the remaining 34·94 per cent. The absence of one digit out of 10 is the commonest event in such trials.

The table may be extended by using the recurrence relation

$$\frac{\Delta^{r+1} 0^{s+1}}{(r+1)!} = (r+1) \frac{\Delta^{r+1} 0^s}{(r+1)!} + \frac{\Delta^r 0^s}{r!}.$$

This equation may be written $$\frac{\Delta^{r+1}0^{s+1}}{(r+1)!}=\frac{\Delta^r(\Delta+1)0^s}{r!}=\frac{\Delta^r 1^s}{r!}$$

showing that the entries derived from powers of natural numbers commencing at *unity*, may be read from this same table, with r and s both increased by one.

Table XXIII. Orthogonal Polynomials

If a polynomial regression line

$$Y=a+bx+cx^2+dx^3+\ldots$$

is to be fitted to a series of n' observations at equal intervals of x, we may conveniently fit instead the equivalent line

$$Y=A+B\xi_1+C\xi_2+D\xi_3+\ldots,$$

where $\xi_1, \xi_2, \xi_3, \ldots$ are orthogonal polynomials of degree 1, 2, 3 . . . ($A=\bar{y}$ and $\xi_1=x-\bar{x}$). A method of determining the coefficients by successive addition, and also of calculating the polynomial values by the same process, is given in *Statistical Methods*. If the numerical values of the ξ's (or multiples $\xi'=\lambda\xi$ of them) are known for each value of x an alternative procedure (set out below) is available. Table XXIII shows these numerical values for all ξ' from ξ'_1 to ξ'_5, and all n' from 3 to 75. In each case the smallest value of λ is chosen which gives integral values to ξ'. This value is shown at the foot of the corresponding column, and is the coefficient of the highest power of x in ξ'. The sum of squares of each set of ξ''s is also shown. Above $n'=8$ only the values for positive $x-\bar{x}$ are given. More extensive tables are available elsewhere (38, 39). The first reference (38) gives values of ξ'_1 to ξ'_5 for all n' to 104, the second (39) gives values of ξ'_1 to ξ'_9 for all n to 52.

The coefficient of each ξ' in the regression equation is found by calculating the sum of the products of the observed values y and the corresponding values of ξ', and dividing by the sum of squares of ξ'. Thus the coefficient of ξ'_2 is

$$C'=\frac{S(y\xi'_2)}{S(\xi'^2_2)}.$$

The corresponding sum of squares in the analysis of variance is

$$C'.S(y\xi'_2)=\frac{\{S(y\xi'_2)\}^2}{S(\xi'^2_2)}$$

and the estimated standard error of C' is $s/\sqrt{S(\xi'^2_2)}$.

If any or all of the values of Y corresponding to the n' observations are required they may be calculated directly from the values of the coefficients and the ξ''s, using the equation

$$Y=A+B'\xi'_1+C'\xi'_2+D'\xi'_3+\ldots$$

This procedure is particularly useful when only a few values are required.

If the regression equation is required in terms of powers of x the formulæ for the ξ's in terms of powers of x given in *Statistical Methods*, or the recurrence formula

$$\xi_{r+1}=\xi_1\xi_r-\frac{r^2(n'^2-r^2)}{4(4r^2-1)}\xi_{r-1}$$

($\xi_0=1$) may be used, together with $\xi'=\lambda\xi$. Alternatively the coefficients of the powers of x in each ξ' may be deduced from the tabulated values of the ξ's, as illustrated in the example below (remembering that the even ξ''s contain only even powers of x, and the odd ξ''s odd powers of x).

The relative advantages of the above method and the alternative process of successive addition depend on the computing facilities available, but with a good multiplying machine, and no printing adding machine, the use of the ξ''s will be found to be decidedly more expeditious, especially when n' is small. If polynomials of the fourth or fifth degree are to be fitted the sums and differences of the pairs of observations (commencing from the centre) should be tabulated, thereby halving the number of multiplications. At the same time the values may frequently be reduced in magnitude by the deduction of a constant amount, and if necessary by rounding off. If polynomials have to be fitted to a number of sets of observations with the same n' it is worth tabulating $\xi'_1+\xi'_3+\xi'_5$ and $\xi'_2+\xi'_4$, and using the checks $Sy(\xi'_1+\xi'_3+\xi'_5)=Sy\,\xi'_1+Sy\,\xi'_3+Sy\,\xi'_5$ and $Sy(\xi'_2+\xi'_4)=Sy\,\xi'_2+Sy\,\xi'_4$.

Example 10

Table 6 shows the yields (bushels per acre) of plots 9 and 7 of Broadbalk wheat field for the 30 years 1855-1884. The only difference in manurial treatment was that plot 9 received nitrate of soda and 7 an equivalent quantity of nitrogen as sulphate of ammonia. Investigate the slow changes in the difference between the two plots by fitting a polynomial of the fifth degree. (This example is discussed in *Statistical Methods*, Sections 26 and 28·1. The values there given have been rounded off to one decimal place.)

TABLE 6

Harvest Year.	9.	7.	9–7.	Harvest Year.	9.	7.	9–7.	Harvest Year.	9.	7.	9–7.
1855	29·6	33·0	−3·4	1865	44·1	40·2	+3·9	1875	30·5	26·6	+3·9
1856	32·4	36·9	−4·5	1866	32·5	29·9	+2·6	1876	33·3	25·5	+7·8
1857	43·8	44·8	−1·0	1867	29·1	22·2	+6·9	1877	40·1	19·1	+21·0
1858	37·6	38·9	−1·3	1868	47·8	39·2	+8·6	1878	37·2	32·2	+5·0
1859	30·0	34·7	−4·7	1869	39·0	28·2	+10·8	1879	21·9	17·2	+4·7
1860	32·6	27·7	+4·9	1870	45·5	41·4	+4·1	1880	34·1	34·3	−·2
1861	33·8	34·9	−1·1	1871	34·4	22·3	+12·1	1881	35·4	26·1	+9·3
1862	43·4	35·9	+7·5	1872	40·7	29·1	+11·6	1882	31·8	34·8	−3·0
1863	55·6	53·7	+1·9	1873	35·8	22·8	+13·0	1883	43·4	36·3	+7·1
1864	51·1	45·8	+5·3	1874	38·2	39·6	−1·4	1884	40·4	37·8	+2·6

The computations are given in Table 7. The first step is to form the sums and differences of pairs of values, working outwards from the centre pair.

Thus $$+14\cdot9 = +4\cdot1+10\cdot8 \text{ and } -6\cdot7 = +4\cdot1-10\cdot8.$$

The totals of the sums and the differences check against the sum and difference of the totals of the first fifteen and last fifteen of the original values. The total sum of squares of the deviations from the mean, 1016·99, can be calculated from the sums and differences, and, as a check, from the original values, at the same time as these totals are obtained.

TABLE 7

Years.	Sum.	Difference.
'70, '69	+14·9	−6·7
'71, '68	+20·7	+3·5
'72, '67	+18·5	+4·7
'73, '66	+15·6	+10·4
'74, '65	+2·5	−5·3
'75, '64	+9·2	−1·4
'76, '63	+9·7	+5·9
'77, '62	+28·5	+13·5
'78, '61	+3·9	+6·1
'79, '60	+9·6	−0·2
'80, '59	−4·9	+4·5
'81, '58	+8·0	+10·6
'82, '57	−4·0	−2·0
'83, '56	+2·6	+11·6
'84, '55	−0·8	+6·0
	+134·0	+61·2
	4·46	

Sums of Products.

Term.	$S\xi' y$.	Regression Coefficient.	Sum of Squares.
1	+1191·6	+0·1325473	157·9434
2	−8990·0	−0·02976190	267·5595
3	−8772·2	$-0\cdot0^{3}4106789$	3·6026
4	+148509·0	$+0\cdot0^{4}4044817$	6·0069
5	+72400·4	$+0\cdot0^{4}3374157$	2·4429

Analysis of Variance.

	Degrees of Freedom.	Sum of Squares.	Mean Square.
Polynomial terms .	5	437·55	87·51
Remainder . .	24	579·44	24·14
Total . .	29	1016·99	

The sums of products with the ξ' values are next calculated, and at the same time the regression coefficients and the sum of squares accounted for by each term. The sums of products may be checked by direct recalculation or by the method described above. Finally the analysis of variance can be completed, giving the residual mean square of 24·14. It will be seen the first two terms account for a substantial part

of the variation, but the mean squares of the remaining three terms are all below the residual mean square. Thus a parabola adequately describes the slow changes.

The equation of the parabola is

$$Y = 4{\cdot}4667 + 0{\cdot}13255\ \xi'_1 - 0{\cdot}029762\ \xi'_2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which can be converted into an equation in x, if this is required, by substitution for ξ'_1 and ξ'_2. If x is measured from the centre point of the series $\xi'_1 = 2x$ and $\xi'_2 = \frac{3}{2}x^2 + C$, the coefficients 2 and $\frac{3}{2}$ of the highest powers of x being given in Table XXIII. Further, since $\xi'_2 = -112$ when $x = +\frac{1}{2}$, we have immediately $C = -112{\cdot}375$.

Alternatively the recurrence formula gives

$$\xi'_2 = \frac{3}{2}\xi_2 = \frac{3}{2}\left[\xi_1 \,.\, \xi_1 - \frac{1^2 \,.\, 899}{4{\cdot}3} \,.\, 1\right] = \frac{3}{2}x^2 - 112{\cdot}375.$$

This gives the equation

$$Y = 7{\cdot}8112 + 0{\cdot}26510x - 0{\cdot}044643x^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The value of Y for any given year can be calculated from (1). Thus for the year 1855.

$$Y = 4{\cdot}4667 + 0{\cdot}13255\ (-29) - 0{\cdot}029762\ (+203) = -5{\cdot}4189.$$

Similarly for the year 1884 the value is $+2{\cdot}2690$. Equation (2) will, of course, give the same values.

The parabola has a maximum at $x = +2{\cdot}969$, *i.e.* at the date 1872·469, Y then having the value $+8{\cdot}2048$. The improvement of plot 9 relative to plot 7 therefore appears to have reached a climax at about this date, there being some evidence of a reduction in the difference subsequently.

As a further exercise polynomials may be fitted to the yields of the two plots separately. The difference of these two polynomials clearly gives the polynomial already determined.

Table XXIV. Calculation of Integrals from equally spaced Ordinates

The table supplies data useful for the mechanical integration of functions calculated at not very closely spaced ordinates. Its use may be illustrated by the series of reciprocals of numbers from 15 to 19. In this case the integral is known to be $\log_e 19/15$, or ·23638,87780,66 ; the average ordinate over the four units in the panel is, therefore, ·05909,71945,16. An accurate calculation can be obtained from the series of reciprocals and their even differences. Taking enough to use the eighth difference, these are as follows to ten places of decimals :—

Number.	Reciprocal.	Differences.			
		2nd.	4th.	6th.	8th.
13	·07692,30769				
14	·07142,85714	73,26008			
15	·06666,66667	59,52380	3,23209		
16	·06250,00000	49,01961	2,33426	28353	
17	·05882,35294	40,84968	1,71996	18438	3745
18	·05555,55556	34,39971	1,29004	12268	
19	·05263,15789	29,23978	98280		
20	·05000,00000	25,06265			
21	·04761,90476				

Using the coefficients for 4 sub-panels in the table for central ordinate and even differences, the successive approximations obtained are as follows :—

	Correction.	Value.	Error.
No difference		·05882,35294,00	−27,36651,16
2nd ,,	+27,23312,00	·05909,58606,00	−13339,16
4th ,,	+13377,47	·05909,71983,47	+38,31
6th ,,	−39,02	·05909,71944,45	−71
8th ,,	+86	·05909,71945,31	+15

To show the adequacy of the adjustments two additional figures have been retained. The evaluation is satisfactory using only 6th differences, although the 8th difference is a four-figure number.

The formulæ using integral multipliers in the first part of the table (4 sub-panels) give exactly equivalent values to the use of 4th and 6th differences.

Tables XXV to XXXII and XXXIV. Logarithms, Trigonometrical Functions, Etc.

These tables comprise the standard functions that are likely to be required by the statistician. Five-figure accuracy has been adopted in most of the tables, since four-figure tables are already sufficiently plentiful, and five figures bridge the gap between the slide-rule (roughly three-figure accuracy) and seven-figure standard tables.

In addition to the ordinary table of logarithms to base 10 of numbers 1·00 to 9·99 (Table XXV) six pages of natural logarithms ($\log_e \cdot 100 - \log_e 99 \cdot 9$) are included (Table XXVI). This table will prove useful to those who find it convenient to work with natural logarithms. Exact squares of all three-figure numbers (Table XXVII) will enable an occasional analysis of variance to be performed without a machine. Square roots (Table XXVIII) are so tabulated that the square roots of each three-figure number and of ten times that number appear together. This facilitates reference to the table, and serves as a reminder of the necessity of considering the position of the decimal point before taking out a square root. Reciprocals (Table XXIX) are given to six figures, since interpolation is not likely to be required in this table when a machine is available, and the additional figure is useful when division is being performed on a machine by means of reciprocals.

Table XXX gives the factorials of the numbers 1-300 to six significant figures, and their logarithms to seven decimal places. Of the trigonometrical functions only natural sines (Table XXXI) and tangents (Table XXXII) have been included. By tabulating $(90° - \theta) \tan \theta$ instead of $\tan \theta$ for values of θ greater than 60° five-figure accuracy of interpolated values is assured, even for high angles.

In Table XXXIV will be found a collection of constants, conversion factors, etc. Only the English, American and metric systems of weights and measures have been included, the more uncommon measures of these systems being given in their most easily remembered forms.

The tabulation of the above tables to 5 and 6 figure accuracy does not preclude their use to four-figure accuracy only when the circumstances warrant it. For this purpose they will be found little slower than the ordinary four-figure tables, since the amount of page turning is the same. When using logarithms to four-figure accuracy it will frequently be found most expeditious, as well as slightly more accurate, to retain 5 figures in the logarithmic calculations, without making any attempt to interpolate to more than four-figure accuracy. General instructions for interpolation are given in the last section of the Introduction.

Table XXXIII. Random Numbers

This table contains 750 two-figure random numbers arranged on six pages. The method of construction of the table, and the tests made on it, are described in the first edition. Methods of using the table may best be made clear by a couple of examples.

Example 11. Required to select a series of numbers at random from the numbers 1-16.

The simplest way of doing this is to select a row, column or diagonal of two-figure numbers (such selection being made without previous inspection of the numbers themselves), and pick out all numbers between 1 and 16 as they occur. With a number like 16, however, such a procedure is tedious and wasteful of numbers. As an alternative we may divide each two-figure number by 16 and take the remainder (0 counting as 16). It is clear that to give all numbers an equal chance of selection the numbers 97, 98, 99 and 00 must be rejected. A modification of this process, which somewhat lightens the arithmetic, is to use the divisor 20 instead of 16, rejecting any number which gives 00, 17, 18 or 19. Thus 73 would be read as 13.

Example 12. Required to arrange 16 treatments, numbered 1-16, in random order.

We may start by selecting a series of numbers between 1 and 16 by one of the methods of the last example, rejecting any number if it occurs a second time. After half the numbers have been selected, however, the process becomes tedious, since the rejections become more numerous, on the average, than the selections. This difficulty may be overcome by choosing the first number by making a selection between 1 and 16, the second by making a selection between 1 and 15, the third by making a selection between 1 and 14, and so on, the actual number selected at each stage being taken to indicate (by counting from one end of the series) which of the numbers still remaining is to be chosen. This operation can easily be performed by writing down all the numbers (or other symbols) in order and striking out each as it is selected. Thus, taking the fourteenth column of two-figure numbers on page IV, which reads

82, 05, 30, 58, 55, 66, 53, 16, 89, 56, 16, 70, 70, 07, 37,

we obtain on dividing successively by 16, 15, 14, 13 . . . the following series of remainders

2, 5, 2, 6, 7, 11, 3, 7, 1, 7, 4, 5, 2, 1, 1.

We therefore select the second, fifth, second, etc., numbers of all numbers still unselected. This gives

2, 6, 3, 9, 11, 16, 5, 13, 1, 15, 10, 14, 7, 4, 8, 12.

Various combinations of the above methods can also be used. An excellent way of proceeding in Example 12 is to select the first 7 treatments by selecting a number between 1 and 16 by dividing by 20, as suggested in Example 11, rejecting any number which occurs a second time, afterwards completing the selection by division of the subsequent two-figure numbers by decreasing divisors according to the method just given. Using this process on the numbers printed above we obtain as the first 7 treatments

2, 5, 10, 18 (rejected), 15, 6, 13, 16.

Division of the subsequent numbers by 9, 8, 7, etc., gives the series 8, 8, 2, etc., so that the corresponding treatments are the eighth, eighth, second, etc., of those remaining, *i.e.* 12, 14, 3, etc.

Interpolation

Many of the tables are provided with special aids to rapid interpolation, such as proportional parts, mean differences, and tabular differences printed in the tables. In other cases linear interpolation is generally sufficient. Thus if the table gives values for u_x and u_{x+1}, and the value is required for $u_{x+\theta}$, we may take

$$u' = \theta u_{x+1} + \phi u_x,$$

where ϕ stands for $1-\theta$. This form is convenient for machine work; mentally it is often quicker to use the differences between the tabular values and calculate

$$u' = u_x + \theta(u_{x+1} - u_x).$$

Modern work, especially that of Jordan (20) and Aitken (18), (19), has much facilitated the use of interpolation, without the aid of differences, in cases for which linear interpolation is thought not to be sufficiently accurate. A very useful improvement on the linear interpolate, based on four instead of two adjacent tabular values, is obtained by supplementing the linear interpolate from the two nearest entries by that from the next two which enclose them. This latter is

$$u'' = \tfrac{1}{3}\{(1+\theta)u_{x+2} + (1+\phi)u_{x-1}\}.$$

Curvature of the entries is shown by u' differing from u'', the error of the former being generally of the same sign as that of the latter, though smaller. To complete the adjustment calculate the product $\theta\phi$, and use

$$u = \tfrac{1}{2}\{(2+\theta\phi)u' - \theta\phi u''\}$$

which is the correct 4-point interpolate, obtained without the use of differences.

In the tables in which mean proportional parts are given for each line of the table, only the proportional parts for 1 to 5 units are included, instead of, as is customary in tables of this type, for 1 to 9 units, it being intended that subtraction from the next higher tabular value should be used when interpolating for fractions above 0·5. The exact procedure will be clear from the examples below. The advantages of this innovation are two-fold: firstly the differences which have to be dealt with are smaller, thus facilitating the arithmetic (which should be done mentally), and secondly the errors arising from the use of *mean* proportional parts for the whole of each line are reduced.

The actual tables in which this type of arrangement has been adopted, for the whole or the greater part of the table, comprise: II (ordinates of the normal distribution), IX (probits), XXV and XXVI (logarithms), XXVIII (square roots), XXXI (sines), and XXXII (tangents). In tables XXV and XXVI (logarithms) the common practice is followed of tabulating the values for the numbers 10-19 on 20 lines instead of 10, so that five values instead of ten occur on each line, thus reducing the range covered by each set of mean proportional parts (which are here changing rather rapidly) to five units of the interval of tabulation.

In Table XXIX (reciprocals) mean differences of consecutive tabular values, instead of mean proportional parts, are given for each line of the table (thus allowing space for the tabulation of six figures instead of five). These mean differences correspond to ten times the mean proportional parts for 1 unit given in the other tables. This course has also been followed for Table VII (transformation of r to z).

In those parts of Table XI (probits) and Table XXIX (reciprocals) where the first differences are changing so rapidly that the mean proportional parts or mean differences are too inaccurate to be of value, the actual first differences between each tabular value and the next are given.

In Table XXVII (squares) no differences are given. Squares of four-figure numbers may be easily derived by the use of the expansion of $(a+b)^2$, as indicated in the note below the table.

In all the above tables, except the last two lines of the table of probits and the first 20 lines (1·0-2·9) of the table of reciprocals, linear interpolation will give results accurate to one unit in the last tabulated place. The

use of mean proportional parts or mean differences will, of course, give errors greater than this in those parts of the tables where the differences are changing rapidly. The actual magnitude of the error may be assessed by seeing how rapidly the proportional parts are changing from line to line, as illustrated in example 13 below. The mean proportional parts should therefore be used with discretion when full tabular accuracy is required.

Example 13

Find the natural logarithm of 11·06. We have $\log_e$ 11·1 = 2·40695. The proportional part for 4 is 357. Thus $\log_e$ 11·06 = 2·40338. The true value = 2·40333,5, the discrepancy being due to the use of the mean proportional part instead of the correct proportional part, which is 0·4 (2·40695—2·39790), *i.e.* 362. As will be seen by inspection the proportional parts are here changing rather rapidly. A closer approximation would have been obtained by taking this change into account, interpolating roughly in the required column of proportional parts, *i.e.* between 357 and 374.

Example 14

Find the square root of 29·426. The square root of 29·4 is 5·4222, and since 2·6 = 3—0·4 the required proportional part is 28—3·7 or 24, giving the value 5·4246. (True value = 5·42457.)

Example 15

Find the angle whose tangent is 0·79134. The nearest tabular value, 0·79259, is that for 38° 24′. The difference is —125. To the nearest minute, therefore, the angle is 38° 21′, a closer approximation being 38° 21·4′. (True value = 38° 21·36′.)

Example 16

Find the angle whose tangent is 2·82. Ordinary inverse interpolation is here impossible, since (90°—θ) tan θ is tabulated, but any approximation may be rapidly improved by dividing the tabular value by the given tangent, and subtracting from 90°. First find a rough approximation by interpolation in the auxiliary marginal table of tangents. This gives a value between 70·4° and 70·5°, *i.e.* between 70° 24′ and 70° 30′.

We then obtain

Angle.	Tabular Value (x).	$x/2{\cdot}82$.	*Plus* Original Angle.
70·4	55·043	19·5188	89·9188
70·5	55·066	19·5270	90·0270

The second column gives the tabular values from the table, the third is the result of dividing these by the given tangent 2·82. Subtracting either of these from 90° gives a better approximation. For full accuracy we use the fourth column, which is the sum of these quotients and the original angles. Inverse linear interpolation between the last two values for the value 90 will now give the required angle, namely 70·4750, *i.e.* 70° 28.50′. (True value = 70° 28·499′.)

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TABLE I. THE NORMAL DISTRIBUTION

P	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09
·0	∞	2·575829	2·326348	2·170090	2·053749	1·959964	1·880794	1·811911	1·750686	1·695398
·1	1·644854	1·598193	1·554774	1·514102	1·475791	1·439521	1·405072	1·372204	1·340755	1·310579
·2	1·281552	1·253565	1·226528	1·200359	1·174987	1·150349	1·126391	1·103063	1·080319	1·058122
·3	1·036433	1·015222	·994458	·974114	·954165	·934589	·915365	·896473	·877896	·859617
·4	·841621	·823894	·806421	·789192	·772193	·755415	·738847	·722479	·706303	·690309
·5	·674490	·658838	·643345	·628006	·612813	·597760	·582841	·568051	·553385	·538836
·6	·524401	·510073	·495850	·481727	·467699	·453762	·439913	·426148	·412463	·398855
·7	·385320	·371856	·358459	·345125	·331853	·318639	·305481	·292375	·279319	·266311
·8	·253347	·240426	·227545	·214702	·201893	·189118	·176374	·163658	·150969	·138304
·9	·125661	·113039	·100434	·087845	·075270	·062707	·050154	·037608	·025069	·012533

P	·001	·000,1	·000,01	·000,001	·000,000,1	·000,000,01	·000,000,001
x	3·29053	3·89059	4·41717	4·89164	5·32672	5·73073	6·10941

The value of P for each entry is found by adding the column heading to the value in the left-hand margin. The corresponding value of x is the deviation such that the probability of an observation falling outside the range from $-x$ to $+x$ is P. For example, P = ·03 for $x = 2·170090$; so that 3 per cent. of normally distributed values will have positive or negative deviations exceeding the standard deviation in the ratio 2·170090 at least.

The table of probits (Table IX) provides a more extensive (P, x) table. The probit table refers to a single tail of the distribution, and the P's derived from that table must therefore be multiplied by 2 to bring them into line with the P's of Table I.

TABLE II. ORDINATES OF THE NORMAL DISTRIBUTION

x	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	1	2	3	4	5
0·0	·3989	·3989	·3989	·3988	·3986	·3984	·3982	·3980	·3977	·3973	0	0	−1	−1	−1
0·1	·3970	·3965	·3961	·3956	·3951	·3945	·3939	·3932	·3925	·3918	−1	−1	−2	−2	−3
0·2	·3910	·3902	·3894	·3885	·3876	·3867	·3857	·3847	·3836	·3825	−1	−2	−3	−4	−5
0·3	·3814	·3802	·3790	·3778	·3765	·3752	·3739	·3725	·3712	·3697	−1	−3	−4	−5	−6
0·4	·3683	·3668	·3653	·3637	·3621	·3605	·3589	·3572	·3555	·3538	−2	−3	−5	−6	−8
0·5	·3521	·3503	·3485	·3467	·3448	·3429	·3410	·3391	·3372	·3352	−2	−4	−6	−8	−9
0·6	·3332	·3312	·3292	·3271	·3251	·3230	·3209	·3187	·3166	·3144	−2	−4	−6	−8	−10
0·7	·3123	·3101	·3079	·3056	·3034	·3011	·2989	·2966	·2943	·2920	−2	−5	−7	−9	−11
0·8	·2897	·2874	·2850	·2827	·2803	·2780	·2756	·2732	·2709	·2685	−2	−5	−7	−9	−12
0·9	·2661	·2637	·2613	·2589	·2565	·2541	·2516	·2492	·2468	·2444	−2	−5	−7	−10	−12
1·0	·2420	·2396	·2371	·2347	·2323	·2299	·2275	·2251	·2227	·2203	−2	−5	−7	−10	−12
1·1	·2179	·2155	·2131	·2107	·2083	·2059	·2036	·2012	·1989	·1965	−2	−5	−7	−10	−12
1·2	·1942	·1919	·1895	·1872	·1849	·1826	·1804	·1781	·1758	·1736	−2	−5	−7	−9	−11
1·3	·1714	·1691	·1669	·1647	·1626	·1604	·1582	·1561	·1539	·1518	−2	−4	−7	−9	−11
1·4	·1497	·1476	·1456	·1435	·1415	·1394	·1374	·1354	·1334	·1315	−2	−4	−6	−8	−10
1·5	·1295	·1276	·1257	·1238	·1219	·1200	·1182	·1163	·1145	·1127	−2	−4	−6	−7	−9
1·6	·1109	·1092	·1074	·1057	·1040	·1023	·1006	·0989	·0973	·0957	−2	−3	−5	−7	−8
1·7	·0940	·0925	·0909	·0893	·0878	·0863	·0848	·0833	·0818	·0804	−2	−3	−5	−6	−8
1·8	·0790	·0775	·0761	·0748	·0734	·0721	·0707	·0694	·0681	·0669	−1	−3	−4	−5	−7
1·9	·0656	·0644	·0632	·0620	·0608	·0596	·0584	·0573	·0562	·0551	−1	−2	−4	−5	−6
2·0	·0540	·0529	·0519	·0508	·0498	·0488	·0478	·0468	·0459	·0449	−1	−2	−3	−4	−5
2·1	·0440	·0431	·0422	·0413	·0404	·0396	·0387	·0379	·0371	·0363	−1	−2	−3	−3	−4
2·2	·0355	·0347	·0339	·0332	·0325	·0317	·0310	·0303	·0297	·0290	−1	−1	−2	−3	−4
2·3	·0283	·0277	·0270	·0264	·0258	·0252	·0246	·0241	·0235	·0229	−1	−1	−2	−2	−3
2·4	·0224	·0219	·0213	·0208	·0203	·0198	·0194	·0189	·0184	·0180	0	−1	−1	−2	−2
2·5	·0175	·0171	·0167	·0163	·0158	·0154	·0151	·0147	·0143	·0139	0	−1	−1	−2	−2
2·6	·0136	·0132	·0129	·0126	·0122	·0119	·0116	·0113	·0110	·0107	0	−1	−1	−1	−2
2·7	·0104	·0101	·0099	·0096	·0093	·0091	·0088	·0086	·0084	·0081	0	−1	−1	−1	−1
2·8	·0079	·0077	·0075	·0073	·0071	·0069	·0067	·0065	·0063	·0061	0	0	−1	−1	−1
2·9	·0060	·0058	·0056	·0055	·0053	·0051	·0050	·0048	·0047	·0046	0	0	0	−1	−1

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
3·0	·0044	·0033	·0024	·0017	·0012	·0009	·0006	·0004	·0003	·0002

TABLE III. DISTRIBUTION OF t

	Probability.												
n	·9	·8	·7	·6	·5	·4	·3	·2	·1	·05	·02	·01	·001
1	·158	·325	·510	·727	1·000	1·376	1·963	3·078	6·314	12·706	31·821	63·657	636·619
2	·142	·289	·445	·617	·816	1·061	1·386	1·886	2·920	4·303	6·965	9·925	31·598
3	·137	·277	·424	·584	·765	·978	1·250	1·638	2·353	3·182	4·541	5·841	12·941
4	·134	·271	·414	·569	·741	·941	1·190	1·533	2·132	2·776	3·747	4·604	8·610
5	·132	·267	·408	·559	·727	·920	1·156	1·476	2·015	2·571	3·365	4·032	6·859
6	·131	·265	·404	·553	·718	·906	1·134	1·440	1·943	2·447	3·143	3·707	5·959
7	·130	·263	·402	·549	·711	·896	1·119	1·415	1·895	2·365	2·998	3·499	5·405
8	·130	·262	·399	·546	·706	·889	1·108	1·397	1·860	2·306	2·896	3·355	5·041
9	·129	·261	·398	·543	·703	·883	1·100	1·383	1·833	2·262	2·821	3·250	4·781
10	·129	·260	·397	·542	·700	·879	1·093	1·372	1·812	2·228	2·764	3·169	4·587
11	·129	·260	·396	·540	·697	·876	1·088	1·363	1·796	2·201	2·718	3·106	4·437
12	·128	·259	·395	·539	·695	·873	1·083	1·356	1·782	2·179	2·681	3·055	4·318
13	·128	·259	·394	·538	·694	·870	1·079	1·350	1·771	2·160	2·650	3·012	4·221
14	·128	·258	·393	·537	·692	·868	1·076	1·345	1·761	2·145	2·624	2·977	4·140
15	·128	·258	·393	·536	·691	·866	1·074	1·341	1·753	2·131	2·602	2·947	4·073
16	·128	·258	·392	·535	·690	·865	1·071	1·337	1·746	2·120	2·583	2·921	4·015
17	·128	·257	·392	·534	·689	·863	1·069	1·333	1·740	2·110	2·567	2·898	3·965
18	·127	·257	·392	·534	·688	·862	1·067	1·330	1·734	2·101	2·552	2·878	3·922
19	·127	·257	·391	·533	·688	·861	1·066	1·328	1·729	2·093	2·539	2·861	3·883
20	·127	·257	·391	·533	·687	·860	1·064	1·325	1·725	2·086	2·528	2·845	3·850
21	·127	·257	·391	·532	·686	·859	1·063	1·323	1·721	2·080	2·518	2·831	3·819
22	·127	·256	·390	·532	·686	·858	1·061	1·321	1·717	2·074	2·508	2·819	3·792
23	·127	·256	·390	·532	·685	·858	1·060	1·319	1·714	2·069	2·500	2·807	3·767
24	·127	·256	·390	·531	·685	·857	1·059	1·318	1·711	2·064	2·492	2·797	3·745
25	·127	·256	·390	·531	·684	·856	1·058	1·316	1·708	2·060	2·485	2·787	3·725
26	·127	·256	·390	·531	·684	·856	1·058	1·315	1·706	2·056	2·479	2·779	3·707
27	·127	·256	·389	·531	·684	·855	1·057	1·314	1·703	2·052	2·473	2·771	3·690
28	·127	·256	·389	·530	·683	·855	1·056	1·313	1·701	2·048	2·467	2·763	3·674
29	·127	·256	·389	·530	·683	·854	1·055	1·311	1·699	2·045	2·462	2·756	3·659
30	·127	·256	·389	·530	·683	·854	1·055	1·310	1·697	2·042	2·457	2·750	3·646
40	·126	·255	·388	·529	·681	·851	1·050	1·303	1·684	2·021	2·423	2·704	3·551
60	·126	·254	·387	·527	·679	·848	1·046	1·296	1·671	2·000	2·390	2·660	3·460
120	·126	·254	·386	·526	·677	·845	1·041	1·289	1·658	1·980	2·358	2·617	3·373
∞	·126	·253	·385	·524	·674	·842	1·036	1·282	1·645	1·960	2·326	2·576	3·291

TABLE IV. DISTRIBUTION OF χ^2

	Probability.												
	·99	·98	·95	·90	·80	·70	·50	·30	·20	·10	·05	·02	·01	·001
1	$\cdot 0^3157$	$\cdot 0^3628$	·00393	·0158	·0642	·148	·455	1·074	1·642	2·706	3·841	5·412	6·635	10·827
2	·0201	·0404	·103	·211	·446	·713	1·386	2·408	3·219	4·605	5·991	7·824	9·210	13·815
3	·115	·185	·352	·584	1·005	1·424	2·366	3·665	4·642	6·251	7·815	9·837	11·345	16·268
4	·297	·429	·711	1·064	1·649	2·195	3·357	4·878	5·989	7·779	9·488	11·668	13·277	18·465
5	·554	·752	1·145	1·610	2·343	3·000	4·351	6·064	7·289	9·236	11·070	13·388	15·086	20·517
6	·872	1·134	1·635	2·204	3·070	3·828	5·348	7·231	8·558	10·645	12·592	15·033	16·812	22·457
7	1·239	1·564	2·167	2·833	3·822	4·671	6·346	8·383	9·803	12·017	14·067	16·622	18·475	24·322
8	1·646	2·032	2·733	3·490	4·594	5·527	7·344	9·524	11·030	13·362	15·507	18·168	20·090	26·125
9	2·088	2·532	3·325	4·168	5·380	6·393	8·343	10·656	12·242	14·684	16·919	19·679	21·666	27·877
10	2·558	3·059	3·940	4·865	6·179	7·267	9·342	11·781	13·442	15·987	18·307	21·161	23·209	29·588
11	3·053	3·609	4·575	5·578	6·989	8·148	10·341	12·899	14·631	17·275	19·675	22·618	24·725	31·264
12	3·571	4·178	5·226	6·304	7·807	9·034	11·340	14·011	15·812	18·549	21·026	24·054	26·217	32·909
13	4·107	4·765	5·892	7·042	8·634	9·926	12·340	15·119	16·985	19·812	22·362	25·472	27·688	34·528
14	4·660	5·368	6·571	7·790	9·467	10·821	13·339	16·222	18·151	21·064	23·685	26·873	29·141	36·123
15	5·229	5·985	7·261	8·547	10·307	11·721	14·339	17·322	19·311	22·307	24·996	28·259	30·578	37·697
16	5·812	6·614	7·962	9·312	11·152	12·624	15·338	18·418	20·465	23·542	26·296	29·633	32·000	39·252
17	6·408	7·255	8·672	10·085	12·002	13·531	16·338	19·511	21·615	24·769	27·587	30·995	33·409	40·790
18	7·015	7·906	9·390	10·865	12·857	14·440	17·338	20·601	22·760	25·989	28·869	32·346	34·805	42·312
19	7·633	8·567	10·117	11·651	13·716	15·352	18·338	21·689	23·900	27·204	30·144	33·687	36·191	43·820
20	8·260	9·237	10·851	12·443	14·578	16·266	19·337	22·775	25·038	28·412	31·410	35·020	37·566	45·315
21	8·897	9·915	11·591	13·240	15·445	17·182	20·337	23·858	26·171	29·615	32·671	36·343	38·932	46·797
22	9·542	10·600	12·338	14·041	16·314	18·101	21·337	24·939	27·301	30·813	33·924	37·659	40·289	48·268
23	10·196	11·293	13·091	14·848	17·187	19·021	22·337	26·018	28·429	32·007	35·172	38·968	41·638	49·728
24	10·856	11·992	13·848	15·659	18·062	19·943	23·337	27·096	29·553	33·196	36·415	40·270	42·980	51·179
25	11·524	12·697	14·611	16·473	18·940	20·867	24·337	28·172	30·675	34·382	37·652	41·566	44·314	52·620
26	12·198	13·409	15·379	17·292	19·820	21·792	25·336	29·246	31·795	35·563	38·885	42·856	45·642	54·052
27	12·879	14·125	16·151	18·114	20·703	22·719	26·336	30·319	32·912	36·741	40·113	44·140	46·963	55·476
28	13·565	14·847	16·928	18·939	21·588	23·647	27·336	31·391	34·027	37·916	41·337	45·419	48·278	56·893
29	14·256	15·574	17·708	19·768	22·475	24·577	28·336	32·461	35·139	39·087	42·557	46·693	49·588	58·302
30	14·953	16·306	18·493	20·599	23·364	25·508	29·336	33·530	36·250	40·256	43·773	47·962	50·892	59·703

For larger values of n, the expression $\sqrt{2\chi^2}-\sqrt{2n-1}$ may be used as a normal deviate with unit variance, remembering that the probability for χ^2 corresponds with that of a single tail of the normal curve.

TABLE V. DISTRIBUTION OF z

20 Per Cent. Points (H. W. Norton)

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	1·1242	1·2425	1·2849	1·3097	1·3198	1·3286	1·3400	1·3508	1·3619	1·3730
2	·6343	·6931	·7125	·7219	·7275	·7313	·7360	·7406	·7452	·7500
3	·4933	·5299	·5385	·5418	·5435	·5444	·5453	·5461	·5465	·5467
4	·4274	·4526	·4551	·4547	·4537	·4528	·4512	·4491	·4464	·4431
5	·3892	·4075	·4062	·4032	·4004	·3982	·3947	·3905	·3853	·3791
6	·3645	·3780	·3740	·3692	·3651	·3618	·3569	·3510	·3438	·3350
7	·3471	·3573	·3512	·3450	·3399	·3358	·3298	·3224	·3135	·3025
8	·3342	·3419	·3342	·3269	·3211	·3163	·3093	·3008	·2904	·2774
9	·3243	·3300	·3211	·3130	·3064	·3011	·2933	·2838	·2721	·2573
10	·3164	·3206	·3107	·3018	·2946	·2889	·2805	·2701	·2572	·2407
11	·3100	·3129	·3021	·2926	·2850	·2790	·2699	·2588	·2449	·2268
12	·3047	·3065	·2951	·2850	·2770	·2706	·2611	·2493	·2345	·2149
13	·3002	·3011	·2891	·2786	·2703	·2636	·2536	·2412	·2255	·2046
14	·2964	·2965	·2840	·2732	·2644	·2575	·2471	·2342	·2178	·1956
15	·2931	·2926	·2795	·2684	·2594	·2522	·2415	·2281	·2110	·1876
16	·2903	·2891	·2757	·2642	·2550	·2476	·2366	·2228	·2050	·1805
17	·2877	·2860	·2723	·2605	·2511	·2436	·2322	·2180	·1997	·1741
18	·2855	·2833	·2692	·2572	·2476	·2399	·2283	·2138	·1949	·1682
19	·2835	·2809	·2665	·2543	·2445	·2367	·2249	·2100	·1905	·1629
20	·2817	·2787	·2641	·2517	·2418	·2338	·2217	·2066	·1866	·1581
21	·2800	·2767	·2619	·2493	·2392	·2311	·2189	·2034	·1829	·1536
22	·2786	·2750	·2599	·2471	·2369	·2287	·2163	·2006	·1798	·1495
23	·2772	·2733	·2581	·2452	·2348	·2265	·2139	·1980	·1767	·1457
24	·2760	·2719	·2564	·2433	·2329	·2245	·2117	·1956	·1740	·1421
25	·2749	·2705	·2549	·2417	·2311	·2226	·2097	·1933	·1715	·1388
26	·2738	·2692	·2534	·2401	·2295	·2209	·2079	·1913	·1691	·1357
27	·2729	·2681	·2521	·2387	·2280	·2193	·2062	·1894	·1669	·1328
28	·2720	·2670	·2509	·2374	·2266	·2178	·2046	·1876	·1648	·1300
29	·2711	·2660	·2498	·2362	·2253	·2165	·2031	·1860	·1629	·1274
30	·2703	·2650	·2487	·2350	·2241	·2152	·2017	·1844	·1611	·1250
40	·2647	·2582	·2411	·2267	·2152	·2058	·1915	·1731	·1477	·1062
60	·2592	·2514	·2334	·2184	·2063	·1965	·1813	·1616	·1338	·0848
120	·2536	·2446	·2258	·2101	·1974	·1870	·1710	·1499	·1193	·0582
∞	·2481	·2379	·2183	·2018	·1885	·1776	·1606	·1379	·1041	0

For high values of n_1 and n_2, z (20 per cent.) $= \dfrac{0{\cdot}8416}{\sqrt{h-0{\cdot}6}} - 0{\cdot}4514\left(\dfrac{1}{n_1}-\dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1} + \dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of n_1 and n_2 are taken.

TABLE V. VARIANCE RATIO

20 Per Cent. Points of e^{2z} (H. W. Norton)

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	9·47	12·00	13·06	13·73	14·01	14·26	14·59	14·90	15·24	15·58
2	3·56	4·00	4·16	4·24	4·28	4·32	4·36	4·40	4·44	4·48
3	2·68	2·89	2·94	2·96	2·97	2·97	2·98	2·98	2·98	2·98
4	2·35	2·47	2·48	2·48	2·48	2·47	2·47	2·46	2·44	2·43
5	2·18	2·26	2·25	2·24	2·23	2·22	2·20	2·18	2·16	2·13
6	2·07	2·13	2·11	2·09	2·08	2·06	2·04	2·02	1·99	1·95
7	2·00	2·04	2·02	1·99	1·97	1·96	1·93	1·91	1·87	1·83
8	1·95	1·98	1·95	1·92	1·90	1·88	1·86	1·83	1·79	1·74
9	1·91	1·94	1·90	1·87	1·85	1·83	1·80	1·76	1·72	1·67
10	1·88	1·90	1·86	1·83	1·80	1·78	1·75	1·72	1·67	1·62
11	1·86	1·87	1·83	1·80	1·77	1·75	1·72	1·68	1·63	1·57
12	1·84	1·85	1·80	1·77	1·74	1·72	1·69	1·65	1·60	1·54
13	1·82	1·83	1·78	1·75	1·72	1·69	1·66	1·62	1·57	1·51
14	1·81	1·81	1·76	1·73	1·70	1·67	1·64	1·60	1·55	1·48
15	1·80	1·79	1·75	1·71	1·68	1·66	1·62	1·58	1·53	1·46
16	1·79	1·78	1·74	1·70	1·67	1·64	1·61	1·56	1·51	1·43
17	1·78	1·77	1·72	1·68	1·65	1·63	1·59	1·55	1·49	1·42
18	1·77	1·76	1·71	1·67	1·64	1·62	1·58	1·53	1·48	1·40
19	1·76	1·75	1·70	1·66	1·63	1·61	1·57	1·52	1·46	1·39
20	1·76	1·75	1·70	1·65	1·62	1·60	1·56	1·51	1·45	1·37
21	1·75	1·74	1·69	1·65	1·61	1·59	1·55	1·50	1·44	1·36
22	1·75	1·73	1·68	1·64	1·61	1·58	1·54	1·49	1·43	1·35
23	1·74	1·73	1·68	1·63	1·60	1·57	1·53	1·49	1·42	1·34
24	1·74	1·72	1·67	1·63	1·59	1·57	1·53	1·48	1·42	1·33
25	1·73	1·72	1·66	1·62	1·59	1·56	1·52	1·47	1·41	1·32
26	1·73	1·71	1·66	1·62	1·58	1·56	1·52	1·47	1·40	1·31
27	1·73	1·71	1·66	1·61	1·58	1·55	1·51	1·46	1·40	1·30
28	1·72	1·71	1·65	1·61	1·57	1·55	1·51	1·46	1·39	1·30
29	1·72	1·70	1·65	1·60	1·57	1·54	1·50	1·45	1·39	1·29
30	1·72	1·70	1·64	1·60	1·57	1·54	1·50	1·45	1·38	1·28
40	1·70	1·68	1·62	1·57	1·54	1·51	1·47	1·41	1·34	1·24
60	1·68	1·65	1·59	1·55	1·51	1·48	1·44	1·38	1·31	1·18
120	1·66	1·63	1·57	1·52	1·48	1·45	1·41	1·35	1·27	1·12
∞	1·64	1·61	1·55	1·50	1·46	1·43	1·38	1·32	1·23	1·00

Lower 20 per cent. points are found by interchange of n_1 and n_2, *i.e.* n_1 must always correspond with the greater mean square.

TABLE V. DISTRIBUTION OF *z*—*continued*

10 Per Cent. Points

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	1·8427	1·9510	1·9907	2·0112	2·0236	2·0320	2·0425	2·0530	2·0636	2·0742
2	1·0716	1·0986	1·1075	1·1120	1·1146	1·1164	1·1186	1·1208	1·1230	1·1252
3	·8558	·8489	·8423	·8379	·8347	·8324	·8293	·8258	·8221	·8179
4	·7570	·7322	·7164	·7064	·6994	·6944	·6875	·6799	·6716	·6623
5	·7006	·6648	·6432	·6293	·6196	·6125	·6029	·5921	·5801	·5665
6	·6643	·6211	·5953	·5786	·5669	·5583	·5465	·5332	·5181	·5007
7	·6390	·5905	·5615	·5427	·5295	·5197	·5061	·4907	·4730	·4523
8	·6203	·5678	·5364	·5160	·5015	·4907	·4757	·4585	·4386	·4148
9	·6060	·5504	·5171	·4953	·4798	·4682	·4520	·4333	·4114	·3849
10	·5947	·5366	·5017	·4788	·4624	·4502	·4330	·4130	·3893	·3602
11	·5855	·5253	·4892	·4653	·4483	·4355	·4173	·3962	·3710	·3395
12	·5779	·5160	·4788	·4541	·4365	·4231	·4043	·3821	·3555	·3219
13	·5715	·5082	·4701	·4447	·4265	·4127	·3932	·3702	·3422	·3066
14	·5661	·5015	·4626	·4366	·4180	·4038	·3836	·3598	·3308	·2931
15	·5614	·4957	·4561	·4296	·4106	·3961	·3754	·3508	·3207	·2813
16	·5572	·4907	·4504	·4235	·4041	·3893	·3681	·3429	·3118	·2706
17	·5537	·4863	·4455	·4181	·3984	·3833	·3617	·3359	·3038	·2611
18	·5505	·4823	·4411	·4134	·3933	·3780	·3560	·3296	·2967	·2524
19	·5476	·4788	·4371	·4091	·3887	·3732	·3508	·3240	·2904	·2445
20	·5451	·4757	·4336	·4052	·3846	·3689	·3462	·3189	·2846	·2373
21	·5427	·4728	·4304	·4017	·3809	·3650	·3420	·3143	·2793	·2307
22	·5407	·4703	·4275	·3986	·3776	·3615	·3382	·3101	·2744	·2245
23	·5388	·4679	·4248	·3957	·3745	·3582	·3347	·3062	·2700	·2188
24	·5370	·4657	·4224	·3931	·3717	·3553	·3315	·3027	·2659	·2135
25	·5354	·4638	·4201	·3906	·3691	·3526	·3286	·2994	·2621	·2086
26	·5339	·4620	·4181	·3884	·3667	·3500	·3258	·2964	·2585	·2039
27	·5326	·4603	·4162	·3863	·3645	·3477	·3233	·2936	·2553	·1996
28	·5313	·4587	·4144	·3844	·3624	·3455	·3210	·2910	·2522	·1955
29	·5301	·4572	·4128	·3826	·3605	·3435	·3188	·2885	·2493	·1916
30	·5290	·4559	·4112	·3809	·3587	·3416	·3167	·2863	·2466	·1880
40	·5211	·4461	·4001	·3688	·3458	·3280	·3019	·2696	·2268	·1599
60	·5133	·4363	·3891	·3567	·3328	·3142	·2868	·2526	·2063	·1279
120	·5054	·4266	·3781	·3446	·3198	·3005	·2717	·2354	·1848	·0081
∞	·4976	·4170	·3671	·3326	·3069	·2866	·2565	·2178	·1622	0

For high values of n_1 and n_2, z (10 per cent.) $= \dfrac{1{\cdot}2816}{\sqrt{h}-0{\cdot}8} - 0{\cdot}6071\left(\dfrac{1}{n_1}-\dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h}=\dfrac{1}{n_1}+\dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of n_1 and n_2 are taken.

TABLE V. VARIANCE RATIO—*continued*

10 Per Cent. Points of e^{2z}

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	39·86	49·50	53·59	55·83	57·24	58·20	59·44	60·70	62·00	63·33
2	8·53	9·00	9·16	9·24	9·29	9·33	9·37	9·41	9·45	9·49
3	5·54	5·46	5·39	5·34	5·31	5·28	5·25	5·22	5·18	5·13
4	4·54	4·32	4·19	4·11	4·05	4·01	3·95	3·90	3·83	3·76
5	4·06	3·78	3·62	3·52	3·45	3·40	3·34	3·27	3·19	3·10
6	3·78	3·46	3·29	3·18	3·11	3·05	2·98	2·90	2·82	2·72
7	3·59	3·26	3·07	2·96	2·88	2·83	2·75	2·67	2·58	2·47
8	3·46	3·11	2·92	2·81	2·73	2·67	2·59	2·50	2·40	2·29
9	3·36	3·01	2·81	2·69	2·61	2·55	2·47	2·38	2·28	2·16
10	3·28	2·92	2·73	2·61	2·52	2·46	2·38	2·28	2·18	2·06
11	3·23	2·86	2·66	2·54	2·45	2·39	2·30	2·21	2·10	1·97
12	3·18	2·81	2·61	2·48	2·39	2·33	2·24	2·15	2·04	1·90
13	3·14	2·76	2·56	2·43	2·35	2·28	2·20	2·10	1·98	1·85
14	3·10	2·73	2·52	2·39	2·31	2·24	2·15	2·05	1·94	1·80
15	3·07	2·70	2·49	2·36	2·27	2·21	2·12	2·02	1·90	1·76
16	3·05	2·67	2·46	2·33	2·24	2·18	2·09	1·99	1·87	1·72
17	3·03	2·64	2·44	2·31	2·22	2·15	2·06	1·96	1·84	1·69
18	3·01	2·62	2·42	2·29	2·20	2·13	2·04	1·93	1·81	1·66
19	2·99	2·61	2·40	2·27	2·18	2·11	2·02	1·91	1·79	1·63
20	2·97	2·59	2·38	2·25	2·16	2·09	2·00	1·89	1·77	1·61
21	2·96	2·57	2·36	2·23	2·14	2·08	1·98	1·88	1·75	1·59
22	2·95	2·56	2·35	2·22	2·13	2·06	1·97	1·86	1·73	1·57
23	2·94	2·55	2·34	2·21	2·11	2·05	1·95	1·84	1·72	1·55
24	2·93	2·54	2·33	2·19	2·10	2·04	1·94	1·83	1·70	1·53
25	2·92	2·53	2·32	2·18	2·09	2·02	1·93	1·82	1·69	1·52
26	2·91	2·52	2·31	2·17	2·08	2·01	1·92	1·81	1·68	1·50
27	2·90	2·51	2·30	2·17	2·07	2·00	1·91	1·80	1·67	1·49
28	2·89	2·50	2·29	2·16	2·06	2·00	1·90	1·79	1·66	1·48
29	2·89	2·50	2·28	2·15	2·06	1·99	1·89	1·78	1·65	1·47
30	2·88	2·49	2·28	2·14	2·05	1·98	1·88	1·77	1·64	1·46
40	2·84	2·44	2·23	2·09	2·00	1·93	1·83	1·71	1·57	1·38
60	2·79	2·39	2·18	2·04	1·95	1·87	1·77	1·66	1·51	1·29
120	2·75	2·35	2·13	1·99	1·90	1·82	1·72	1·60	1·45	1·19
∞	2·71	2·30	2·08	1·94	1·85	1·77	1·67	1·55	1·38	1·00

Lower 10 per cent. points are found by interchange of n_1 and n_2, *i.e.* n_1 must always correspond with the greater mean square.

TABLE V. DISTRIBUTION OF z—*continued*

5 Per Cent. Points

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	2·5421	2·6479	2·6870	2·7071	2·7194	2·7276	2·7380	2·7484	2·7588	2·7693
2	1·4592	1·4722	1·4765	1·4787	1·4800	1·4808	1·4819	1·4830	1·4840	1·4851
3	1·1577	1·1284	1·1137	1·1051	1·0994	1·0953	1·0899	1·0842	1·0781	1·0716
4	1·0212	·9690	·9429	·9272	·9168	·9093	·8993	·8885	·8767	·8639
5	·9441	·8777	·8441	·8236	·8097	·7997	·7862	·7714	·7550	·7368
6	·8948	·8188	·7798	·7558	·7394	·7274	·7112	·6931	·6729	·6499
7	·8606	·7777	·7347	·7080	·6896	·6761	·6576	·6369	·6134	·5862
8	·8355	·7475	·7014	·6725	·6525	·6378	·6175	·5945	·5682	·5371
9	·8163	·7242	·6757	·6450	·6238	·6080	·5862	·5613	·5324	·4979
10	·8012	·7058	·6553	·6232	·6009	·5843	·5611	·5346	·5035	·4657
11	·7889	·6909	·6387	·6055	·5822	·5648	·5406	·5126	·4795	·4387
12	·7788	·6786	·6250	·5907	·5666	·5487	·5234	·4941	·4592	·4156
13	·7703	·6682	·6134	·5783	·5535	·5350	·5089	·4785	·4419	·3957
14	·7630	·6594	·6036	·5677	·5423	·5233	·4964	·4649	·4269	·3782
15	·7568	·6518	·5950	·5585	·5326	·5131	·4855	·4532	·4138	·3628
16	·7514	·6451	·5876	·5505	·5241	·5042	·4760	·4428	·4022	·3490
17	·7466	·6393	·5811	·5434	·5166	·4964	·4676	·4337	·3919	·3366
18	·7424	·6341	·5753	·5371	·5099	·4894	·4602	·4255	·3827	·3253
19	·7386	·6295	·5701	·5315	·5040	·4832	·4535	·4182	·3743	·3151
20	·7352	·6254	·5654	·5265	·4986	·4776	·4474	·4116	·3668	·3057
21	·7322	·6216	·5612	·5219	·4938	·4725	·4420	·4055	·3599	·2971
22	·7294	·6182	·5574	·5178	·4894	·4679	·4370	·4001	·3536	·2892
23	·7269	·6151	·5540	·5140	·4854	·4636	·4325	·3950	·3478	·2818
24	·7246	·6123	·5508	·5106	·4817	·4598	·4283	·3904	·3425	·2749
25	·7225	·6097	·5478	·5074	·4783	·4562	·4244	·3862	·3376	·2685
26	·7205	·6073	·5451	·5045	·4752	·4529	·4209	·3823	·3330	·2625
27	·7187	·6051	·5427	·5017	·4723	·4499	·4176	·3786	·3287	·2569
28	·7171	·6030	·5403	·4992	·4696	·4471	·4146	·3752	·3248	·2516
29	·7155	·6011	·5382	·4969	·4671	·4444	·4117	·3720	·3211	·2466
30	·7141	·5994	·5362	·4947	·4648	·4420	·4090	·3691	·3176	·2419
40	·7037	·5866	·5217	·4789	·4479	·4242	·3897	·3475	·2920	·2057
60	·6933	·5738	·5073	·4632	·4311	·4064	·3702	·3255	·2654	·1644
120	·6830	·5611	·4930	·4475	·4143	·3885	·3506	·3032	·2376	·1131
∞	·6729	·5486	·4787	·4319	·3974	·3706	·3309	·2804	·2085	0

For high values of n_1 and n_2, $z\,(5 \text{ per cent.}) = \dfrac{1{\cdot}6449}{\sqrt{h-1}} - 0{\cdot}7843\left(\dfrac{1}{n_1} - \dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1} + \dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of n_1 and n_2 are taken.

TABLE V. VARIANCE RATIO—*continued*

5 Per Cent. Points of e^{2z}

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	161·4	199·5	215·7	224·6	230·2	234·0	238·9	243·9	249·0	254·3
2	18·51	19·00	19·16	19·25	19·30	19·33	19·37	19·41	19·45	19·50
3	10·13	9·55	9·28	9·12	9·01	8·94	8·84	8·74	8·64	8·53
4	7·71	6·94	6·59	6·39	6·26	6·16	6·04	5·91	5·77	5·63
5	6·61	5·79	5·41	5·19	5·05	4·95	4·82	4·68	4·53	4·36
6	5·99	5·14	4·76	4·53	4·39	4·28	4·15	4·00	3·84	3·67
7	5·59	4·74	4·35	4·12	3·97	3·87	3·73	3·57	3·41	3·23
8	5·32	4·46	4·07	3·84	3·69	3·58	3·44	3·28	3·12	2·93
9	5·12	4·26	3·86	3·63	3·48	3·37	3·23	3·07	2·90	2·71
10	4·96	4·10	3·71	3·48	3·33	3·22	3·07	2·91	2·74	2·54
11	4·84	3·98	3·59	3·36	3·20	3·09	2·95	2·79	2·61	2·40
12	4·75	3·88	3·49	3·26	3·11	3·00	2·85	2·69	2·50	2·30
13	4·67	3·80	3·41	3·18	3·02	2·92	2·77	2·60	2·42	2·21
14	4·60	3·74	3·34	3·11	2·96	2·85	2·70	2·53	2·35	2·13
15	4·54	3·68	3·29	3·06	2·90	2·79	2·64	2·48	2·29	2·07
16	4·49	3·63	3·24	3·01	2·85	2·74	2·59	2·42	2·24	2·01
17	4·45	3·59	3·20	2·96	2·81	2·70	2·55	2·38	2·19	1·96
18	4·41	3·55	3·16	2·93	2·77	2·66	2·51	2·34	2·15	1·92
19	4·38	3·52	3·13	2·90	2·74	2·63	2·48	2·31	2·11	1·88
20	4·35	3·49	3·10	2·87	2·71	2·60	2·45	2·28	2·08	1·84
21	4·32	3·47	3·07	2·84	2·68	2·57	2·42	2·25	2·05	1·81
22	4·30	3·44	3·05	2·82	2·66	2·55	2·40	2·23	2·03	1·78
23	4·28	3·42	3·03	2·80	2·64	2·53	2·38	2·20	2·00	1·76
24	4·26	3·40	3·01	2·78	2·62	2·51	2·36	2·18	1·98	1·73
25	4·24	3·38	2·99	2·76	2·60	2·49	2·34	2·16	1·96	1·71
26	4·22	3·37	2·98	2·74	2·59	2·47	2·32	2·15	1·95	1·69
27	4·21	3·35	2·96	2·73	2·57	2·46	2·30	2·13	1·93	1·67
28	4·20	3·34	2·95	2·71	2·56	2·44	2·29	2·12	1·91	1·65
29	4·18	3·33	2·93	2·70	2·54	2·43	2·28	2·10	1·90	1·64
30	4·17	3·32	2·92	2·69	2·53	2·42	2·27	2·09	1·89	1·62
40	4·08	3·23	2·84	2·61	2·45	2·34	2·18	2·00	1·79	1·51
60	4·00	3·15	2·76	2·52	2·37	2·25	2·10	1·92	1·70	1·39
120	3·92	3·07	2·68	2·45	2·29	2·17	2·02	1·83	1·61	1·25
∞	3·84	2·99	2·60	2·37	2·21	2·09	1·94	1·75	1·52	1·00

Lower 5 per cent. points are found by interchange of n_1 and n_2, *i.e.* n_1 must always correspond with the greater mean square.

TABLE V. DISTRIBUTION OF *z*—*continued*

1 Per Cent. Points

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	4·1535	4·2585	4·2974	4·3175	4·3297	4·3379	4·3482	4·3585	4·3689	4·3794
2	2·2950	2·2976	2·2984	2·2988	2·2991	2·2992	2·2994	2·2997	2·2999	2·3001
3	1·7649	1·7140	1·6915	1·6786	1·6703	1·6645	1·6569	1·6489	1·6404	1·6314
4	1·5270	1·4452	1·4075	1·3856	1·3711	1·3609	1·3473	1·3327	1·3170	1·3000
5	1·3943	1·2929	1·2449	1·2164	1·1974	1·1838	1·1656	1·1457	1·1239	1·0997
6	1·3103	1·1955	1·1401	1·1068	1·0843	1·0680	1·0460	1·0218	·9948	·9643
7	1·2526	1·1281	1·0672	1·0300	1·0048	·9864	·9614	·9335	·9020	·8658
8	1·2106	1·0787	1·0135	·9734	·9459	·9259	·8983	·8673	·8319	·7904
9	1·1786	1·0411	·9724	·9299	·9006	·8791	·8494	·8157	·7769	·7305
10	1·1535	1·0114	·9399	·8954	·8646	·8419	·8104	·7744	·7324	·6816
11	1·1333	·9874	·9136	·8674	·8354	·8116	·7785	·7405	·6958	·6408
12	1·1166	·9677	·8919	·8443	·8111	·7864	·7520	·7122	·6649	·6061
13	1·1027	·9511	·8737	·8248	·7907	·7652	·7295	·6882	·6386	·5761
14	1·0909	·9370	·8581	·8082	·7732	·7471	·7103	·6675	·6159	·5500
15	1·0807	·9249	·8448	·7939	·7582	·7314	·6937	·6496	·5961	·5269
16	1·0719	·9144	·8331	·7814	·7450	·7177	·6791	·6339	·5786	·5064
17	1·0641	·9051	·8229	·7705	·7335	·7057	·6663	·6199	·5630	·4879
18	1·0572	·8970	·8138	·7607	·7232	·6950	·6549	·6075	·5491	·4712
19	1·0511	·8897	·8057	·7521	·7140	·6854	·6447	·5964	·5366	·4560
20	1·0457	·8831	·7985	·7443	·7058	·6768	·6355	·5864	·5253	·4421
21	1·0408	·8772	·7920	·7372	·6984	·6690	·6272	·5773	·5150	·4294
22	1·0363	·8719	·7860	·7309	·6916	·6620	·6196	·5691	·5056	·4176
23	1·0322	·8670	·7806	·7251	·6855	·6555	·6127	·5615	·4969	·4068
24	1·0285	·8626	·7757	·7197	·6799	·6496	·6064	·5545	·4890	·3967
25	1·0251	·8585	·7712	·7148	·6747	·6442	·6006	·5481	·4816	·3872
26	1·0220	·8548	·7670	·7103	·6699	·6392	·5952	·5422	·4748	·3784
27	1·0191	·8513	·7631	·7062	·6655	·6346	·5902	·5367	·4685	·3701
28	1·0164	·8481	·7595	·7023	·6614	·6303	·5856	·5316	·4626	·3624
29	1·0139	·8451	·7562	·6987	·6576	·6263	·5813	·5269	·4570	·3550
30	1·0116	·8423	·7531	·6954	·6540	·6226	·5773	·5224	·4519	·3481
40	·9949	·8223	·7307	·6712	·6283	·5956	·5481	·4901	·4138	·2952
60	·9784	·8025	·7086	·6472	·6028	·5687	·5189	·4574	·3746	·2352
120	·9622	·7829	·6867	·6234	·5774	·5419	·4897	·4243	·3339	·1612
∞	·9462	·7636	·6651	·5999	·5522	·5152	·4604	·3908	·2913	0

For high values of n_1 and n_2, z (1 per cent.) $= \dfrac{2\cdot3263}{\sqrt{h-1\cdot4}} - 1\cdot235\left(\dfrac{1}{n_1}-\dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1} + \dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of n_1 and n_2 are taken.

TABLE V. VARIANCE RATIO—*continued*

1 Per Cent. Points of e^{2z}

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	4052	4999	5403	5625	5764	5859	5981	6106	6234	6366
2	98·49	99·00	99·17	99·25	99·30	99·33	99·36	99·42	99·46	99·50
3	34·12	30·81	29·46	28·71	28·24	27·91	27·49	27·05	26·60	26·12
4	21·20	18·00	16·69	15·98	15·52	15·21	14·80	14·37	13·93	13·46
5	16·26	13·27	12·06	11·39	10·97	10·67	10·29	9·89	9·47	9·02
6	13·74	10·92	9·78	9·15	8·75	8·47	8·10	7·72	7·31	6·88
7	12·25	9·55	8·45	7·85	7·46	7·19	6·84	6·47	6·07	5·65
8	11·26	8·65	7·59	7·01	6·63	6·37	6·03	5·67	5·28	4·86
9	10·56	8·02	6·99	6·42	6·06	5·80	5·47	5·11	4·73	4·31
10	10·04	7·56	6·55	5·99	5·64	5·39	5·06	4·71	4·33	3·91
11	9·65	7·20	6·22	5·67	5·32	5·07	4·74	4·40	4·02	3·60
12	9·33	6·93	5·95	5·41	5·06	4·82	4·50	4·16	3·78	3·36
13	9·07	6·70	5·74	5·20	4·86	4·62	4·30	3·96	3·59	3·16
14	8·86	6·51	5·56	5·03	4·69	4·46	4·14	3·80	3·43	3·00
15	8·68	6·36	5·42	4·89	4·56	4·32	4·00	3·67	3·29	2·87
16	8·53	6·23	5·29	4·77	4·44	4·20	3·89	3·55	3·18	2·75
17	8·40	6·11	5·18	4·67	4·34	4·10	3·79	3·45	3·08	2·65
18	8·28	6·01	5·09	4·58	4·25	4·01	3·71	3·37	3·00	2·57
19	8·18	5·93	5·01	4·50	4·17	3·94	3·63	3·30	2·92	2·49
20	8·10	5·85	4·94	4·43	4·10	3·87	3·56	3·23	2·86	2·42
21	8·02	5·78	4·87	4·37	4·04	3·81	3·51	3·17	2·80	2·36
22	7·94	5·72	4·82	4·31	3·99	3·76	3·45	3·12	2·75	2·31
23	7·88	5·66	4·76	4·26	3·94	3·71	3·41	3·07	2·70	2·26
24	7·82	5·61	4·72	4·22	3·90	3·67	3·36	3·03	2·66	2·21
25	7·77	5·57	4·68	4·18	3·86	3·63	3·32	2·99	2·62	2·17
26	7·72	5·53	4·64	4·14	3·82	3·59	3·29	2·96	2·58	2·13
27	7·68	5·49	4·60	4·11	3·78	3·56	3·26	2·93	2·55	2·10
28	7·64	5·45	4·57	4·07	3·75	3·53	3·23	2·90	2·52	2·06
29	7·60	5·42	4·54	4·04	3·73	3·50	3·20	2·87	2·49	2·03
30	7·56	5·39	4·51	4·02	3·70	3·47	3·17	2·84	2·47	2·01
40	7·31	5·18	4·31	3·83	3·51	3·29	2·99	2·66	2·29	1·80
60	7·08	4·98	4·13	3·65	3·34	3·12	2·82	2·50	2·12	1·60
120	6·85	4·79	3·95	3·48	3·17	2·96	2·66	2·34	1·95	1·38
∞	6·64	4·60	3·78	3·32	3·02	2·80	2·51	2·18	1·79	1·00

Lower 1 per cent. points are found by interchange of n_1 and n_2, *i.e.* n_1 must always correspond with the greater mean square.

TABLE V. DISTRIBUTION OF *z*—*continued*

0·1 Per Cent. Points (Colcord and Deming)

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	6·4562	6·5612	6·5966	6·6201	6·6323	6·6405	6·6508	6·6611	6·6715	6·6819
2	3·4531	3·4534	3·4535	3·4535	3·4535	3·4535	3·4536	3·4536	3·4536	3·4536
3	2·5604	2·5003	2·4748	2·4603	2·4511	2·4446	2·4361	2·4272	2·4179	2·4081
4	2·1529	2·0574	2·0143	1·9892	1·9728	1·9612	1·9459	1·9294	1·9118	1·8927
5	1·9255	1·8002	1·7513	1·7184	1·6964	1·6808	1·6596	1·6370	1·6123	1·5845
6	1·7849	1·6479	1·5828	1·5433	1·5177	1·4986	1·4730	1·4449	1·4134	1·3783
7	1·6874	1·5384	1·4662	1·4221	1·3927	1·3711	1·3417	1·3090	1·2721	1·2296
8	1·6177	1·4587	1·3809	1·3372	1·3008	1·2770	1·2443	1·2077	1·1662	1·1169
9	1·5646	1·3982	1·3160	1·2653	1·2304	1·2047	1·1694	1·1293	1·0830	1·0279
10	1·5232	1·3509	1·2650	1·2116	1·1748	1·1475	1·1098	1·0668	1·0165	·9557
11	1·4900	1·3128	1·2238	1·1683	1·1297	1·1012	1·0614	1·0157	·9619	·8957
12	1·4627	1·2814	1·1900	1·1326	1·0926	1·0628	1·0213	·9733	·9162	·8450
13	1·4400	1·2553	1·1616	1·1026	1·0614	1·0306	·9875	·9374	·8774	·8014
14	1·4208	1·2332	1·1376	1·0772	1·0348	1·0031	·9586	·9066	·8439	·7635
15	1·4043	1·2141	1·1169	1·0553	1·0119	·9795	·9336	·8800	·8147	·7301
16	1·3900	1·1976	1·0989	1·0362	·9920	·9588	·9119	·8567	·7891	·7005
17	1·3775	1·1832	1·0832	1·0195	·9745	·9407	·8927	·8361	·7664	·6740
18	1·3665	1·1704	1·0693	1·0047	·9590	·9246	·8757	·8178	·7462	·6502
19	1·3567	1·1591	1·0569	·9915	·9442	·9103	·8605	·8014	·7277	·6285
20	1·3480	1·1489	1·0458	·9798	·9329	·8974	·8469	·7867	·7115	·6086
21	1·3401	1·1398	1·0358	·9691	·9217	·8858	·8346	·7735	·6964	·5904
22	1·3329	1·1315	1·0268	·9595	·9116	·8753	·8234	·7612	·6828	·5738
23	1·3264	1·1240	1·0186	·9507	·9024	·8657	·8132	·7501	·6704	·5583
24	1·3205	1·1171	1·0111	·9427	·8939	·8569	·8038	·7400	·6589	·5440
25	1·3151	1·1108	1·0041	·9354	·8862	·8489	·7953	·7306	·6483	·5307
26	1·3101	1·1050	·9978	·9286	·8791	·8415	·7873	·7220	·6385	·5183
27	1·3055	1·0997	·9920	·9223	·8725	·8346	·7800	·7140	·6294	·5066
28	1·3013	1·0947	·9866	·9165	·8664	·8282	·7732	·7066	·6209	·4957
29	1·2973	1·0903	·9815	·9112	·8607	·8223	·7679	·6997	·6129	·4853
30	1·2936	1·0859	·9768	·9061	·8554	·8168	·7610	·6932	·6056	·4756
40	1·2674	1·0552	·9435	·8701	·8174	·7771	·7184	·6463	·5513	·4016
60	1·2413	1·0248	·9100	·8345	·7798	·7377	·6760	·5992	·4955	·3198
120	1·2158	·9952	·8773	·7994	·7426	·6986	·6338	·5519	·4380	·2199
∞	1·1910	·9663	·8453	·7648	·7059	·6599	·5917	·5044	·3786	0

For high values of n_1 and n_2, z (0·1 per cent.) $= \dfrac{3\cdot0902}{\sqrt{h-2\cdot1}} - 1\cdot925\left(\dfrac{1}{n_1}-\dfrac{1}{n_2}\right)$ approximately, where $\dfrac{2}{h} = \dfrac{1}{n_1}+\dfrac{1}{n_2}$. In other parts of the table interpolation is approximately linear if reciprocals of n_1 and n_2 are taken.

TABLE V. VARIANCE RATIO—*continued*

0·1 Per Cent. Points of e^{2z}

n_2 \ n_1	1	2	3	4	5	6	8	12	24	∞
1	405284	500000	540379	562500	576405	585937	598144	610667	623497	636619
2	998·5	999·0	999·2	999·2	999·3	999·3	999·4	999·4	999·5	999·5
3	167·5	148·5	141·1	137·1	134·6	132·8	130·6	128·3	125·9	123·5
4	74·14	61·25	56·18	53·44	51·71	50·53	49·00	47·41	45·77	44·05
5	47·04	36·61	33·20	31·09	29·75	28·84	27·64	26·42	25·14	23·78
6	35·51	27·00	23·70	21·90	20·81	20·03	19·03	17·99	16·89	15·75
7	29·22	21·69	18·77	17·19	16·21	15·52	14·63	13·71	12·73	11·69
8	25·42	18·49	15·83	14·39	13·49	12·86	12·04	11·19	10·30	9·34
9	22·86	16·39	13·90	12·56	11·71	11·13	10·37	9·57	8·72	7·81
10	21·04	14·91	12·55	11·28	10·48	9·92	9·20	8·45	7·64	6·76
11	19·69	13·81	11·56	10·35	9·58	9·05	8·35	7·63	6·85	6·00
12	18·64	12·97	10·80	9·63	8·89	8·38	7·71	7·00	6·25	5·42
13	17·81	12·31	10·21	9·07	8·35	7·86	7·21	6·52	5·78	4·97
14	17·14	11·78	9·73	8·62	7·92	7·43	6·80	6·13	5·41	4·60
15	16·59	11·34	9·34	8·25	7·57	7·09	6·47	5·81	5·10	4·31
16	16·12	10·97	9·00	7·94	7·27	6·81	6·19	5·55	4·85	4·06
17	15·72	10·66	8·73	7·68	7·02	6·56	5·96	5·32	4·63	3·85
18	15·38	10·39	8·49	7·46	6·81	6·35	5·76	5·13	4·45	3·67
19	15·08	10·16	8·28	7·26	6·61	6·18	5·59	4·97	4·29	3·52
20	14·82	9·95	8·10	7·10	6·46	6·02	5·44	4·82	4·15	3·38
21	14·59	9·77	7·94	6·95	6·32	5·88	5·31	4·70	4·03	3·26
22	14·38	9·61	7·80	6·81	6·19	5·76	5·19	4·58	3·92	3·15
23	14·19	9·47	7·67	6·69	6·08	5·65	5·09	4·48	3·82	3·05
24	14·03	9·34	7·55	6·59	5·98	5·55	4·99	4·39	3·74	2·97
25	13·88	9·22	7·45	6·49	5·88	5·46	4·91	4·31	3·66	2·89
26	13·74	9·12	7·36	6·41	5·80	5·38	4·83	4·24	3·59	2·82
27	13·61	9·02	7·27	6·33	5·73	5·31	4·76	4·17	3·52	2·75
28	13·50	8·93	7·19	6·25	5·66	5·24	4·69	4·11	3·46	2·70
29	13·39	8·85	7·12	6·19	5·59	5·18	4·64	4·05	3·41	2·64
30	13·29	8·77	7·05	6·12	5·53	5·12	4·58	4·00	3·36	2·59
40	12·61	8·25	6·60	5·70	5·13	4·73	4·21	3·64	3·01	2·23
60	11·97	7·76	6·17	5·31	4·76	4·37	3·87	3·31	2·69	1·90
120	11·38	7·31	5·79	4·95	4·42	4·04	3·55	3·02	2·40	1·56
∞	10·83	6·91	5·42	4·62	4·10	3·74	3·27	2·74	2·13	1·00

Lower 0·1 per cent. points are found by interchange of n_1 and n_2, *i.e.* n_1 must always correspond with the greater mean square.

Table VI. Significance of Difference between Two Means
(P. V. Sukhatme)

	n_1	0°	15°	30°	45°	60°	75°	90°
5 per cent. points	6	2·447	2·440	2·435	2·435	2·435	2·440	2·447
	8	2·447	2·430	2·398	2·364	2·331	2·310	2·306
$n_2 = 6$	12	2·447	2·423	2·367	2·301	2·239	2·193	2·179
	24	2·447	2·418	2·342	2·247	2·156	2·088	2·064
	∞	2·447	2·413	2·322	2·201	2·082	1·993	1·960
	6	2·306	2·310	2·331	2·364	2·398	2·430	2·447
	8	2·306	2·300	2·294	2·292	2·294	2·300	2·306
$n_2 = 8$	12	2·306	2·292	2·262	2·229	2·201	2·183	2·179
	24	2·306	2·286	2·236	2·175	2·118	2·077	2·064
	∞	2·306	2·281	2·215	2·128	2·044	1·982	1·960
	6	2·179	2 193	2·239	2·301	2·367	2·423	2·447
	8	2·179	2 183	2·201	2·229	2·262	2·292	2·306
$n_2 = 12$	12	2·179	2·175	2·169	2·167	2·169	2·175	2·179
	24	2·179	2·168	2·142	2·112	2·085	2·069	2·064
	∞	2·179	2·163	2·120	2·064	2·011	1·973	1·960
	6	2·064	2·088	2·156	2·247	2·342	2·418	2·447
	8	2·064	2·077	2·118	2·175	2·236	2·286	2·306
$n_2 = 24$	12	2·064	2·069	2·085	2·112	2·142	2·168	2·179
	24	2·064	2·062	2·058	2·056	2·058	2·062	2·064
	∞	2·064	2·056	2·035	2·009	1·983	1·966	1·960
	6	1·960	1·993	2·082	2·201	2·322	2·413	2·447
	8	1·960	1·982	2·044	2·128	2·215	2·281	2·306
$n_2 = \infty$	12	1·960	1·973	2·011	2·064	2·120	2·163	2·179
	24	1·960	1·966	1·983	2·009	2·035	2·056	2·064
	∞	1·960	1·960	1·960	1·960	1·960	1·960	1·960
1 per cent. points	6	3·707	3·654	3·557	3·514	3·557	3·654	3·707
	8	3·707	3·643	3·495	3·363	3·307	3·328	3·355
$n_2 = 6$	12	3·707	3·636	3·453	3·246	3·104	3·053	3·055
	24	3·707	3·631	3·424	3·158	2·938	2·822	2·797
	∞	3·707	3·626	3·402	3·093	2·804	2·627	2·576
	6	3·355	3·328	3·307	3·363	3·495	3·643	3·707
	8	3·355	3·316	3·239	3·206	3·239	3·316	3·355
$n_2 = 8$	12	3·355	3·307	3·192	3·083	3·032	3·039	3·055
	24	3·355	3·301	3·158	2·988	2·862	2·805	2·797
	∞	3·355	3·295	3·132	2·916	2·723	2·608	2·576
	6	3·055	3·053	3·104	3·246	3·453	3·636	3·707
	8	3·055	3·039	3·032	3·083	3·192	3·307	3·355
$n_2 = 12$	12	3·055	3·029	2·978	2·954	2·978	3·029	3·055
	24	3·055	3·020	2·938	2·853	2·803	2·793	2·797
	∞	3·055	3·014	2·909	2·775	2·661	2·595	2·576
	6	2·797	2·822	2·938	3·158	3·424	3·631	3·707
	8	2·797	2·805	2·862	2·988	3·158	3·301	3·355
$n_2 = 24$	12	2·797	2·793	2·803	2·853	2·938	3·020	3·055
	24	2·797	2·785	2·759	2·747	2·759	2·785	2·797
	∞	2·797	2·777	2·726	2·664	2·613	2·585	2·576
	6	2·576	2·627	2·804	3·093	3·402	3·626	3·707
	8	2·576	2·608	2·723	2·916	3·132	3·295	3·355
$n_2 = \infty$	12	2·576	2·595	2·661	2·775	2·909	3·014	3·055
	24	2·576	2·585	2·613	2·664	2·726	2·777	2·797
	∞	2·576	2·576	2·576	2·576	2·576	2·576	2·576

TABLE V2. SIGNIFICANCE OF DIFFERENCE BETWEEN TWO MEANS

One component of error distributed normally, the other in Student's distribution

P	n_2	0°(t)	10°	20°	30°	40°	50°	60°	70°	80°	90°(x)
	10	1·812	1·808	1·794	1·774	1·749	1·721	1·693	1·668	1·651	1·645
	12	1·782	1·778	1·767	1·751	1·730	1·707	1·684	1·664	1·650	1·645
	15	1·753	1·750	1·741	1·728	1·711	1·693	1·675	1·659	1·649	1·645
10 per cent.	20	1·725	1·722	1·716	1·706	1·694	1·680	1·667	1·656	1·648	1·645
	30	1·697	1·696	1·692	1·685	1·677	1·668	1·659	1·652	1·647	1·645
	60	1·671	1·670	1·668	1·665	1·661	1·656	1·652	1·648	1·646	1·645
	∞	1·645	1·645	1·645	1·645	1·645	1·645	1·645	1·645	1·645	1·645
	10	2·228	2·219	2·194	2·157	2·112	2·066	2·024	1·989	1·967	1·960
	12	2·179	2·171	2·151	2·120	2·083	2·046	2·011	1·984	1·966	1·960
	15	2·131	2·126	2·109	2·085	2·056	2·026	1·999	1·978	1·965	1·960
5 per cent.	20	2·086	2·082	2·069	2·051	2·030	2·008	1·989	1·973	1·963	1·960
	30	2·042	2·039	2·031	2·019	2·005	1·991	1·978	1·968	1·962	1·960
	60	2·000	1·999	1·995	1·989	1·982	1·975	1·969	1·964	1·961	1·960
	∞	1·960	1·960	1·960	1·960	1·960	1·960	1·960	1·960	1·960	1·960
	10	2·764	2·748	2·704	2·637	2·559	2·481	2·414	2·364	2·335	2·326
	12	2·681	2·668	2·631	2·576	2·513	2·450	2·396	2·356	2·334	2·326
	15	2·602	2·592	2·563	2·520	2·470	2·421	2·379	2·349	2·332	2·326
2 per cent.	20	2·528	2·520	2·498	2·466	2·430	2·394	2·364	2·343	2·330	2·326
	30	2·457	2·452	2·438	2·417	2·393	2·370	2·351	2·337	2·329	2·326
	60	2·390	2·388	2·380	2·370	2·358	2·347	2·338	2·331	2·328	2·326
	∞	2·326	2·326	2·326	2·326	2·326	2·326	2·326	2·326	2·326	2·326
	10	3·169	3·148	3·086	2·993	2·883	2·775	2·684	2·620	2·586	2·576
	12	3·055	3·037	2·985	2·909	2·820	2·733	2·661	2·611	2·584	2·576
	15	2·947	2·932	2·892	2·831	2·762	2·695	2·640	2·603	2·582	2·576
1 per cent.	20	2·845	2·835	2·804	2·760	2·709	2·661	2·622	2·595	2·580	2·576
	30	2·750	2·743	2·723	2·693	2·661	2·630	2·605	2·588	2·579	2·576
	60	2·660	2·657	2·647	2·632	2·616	2·601	2·590	2·582	2·577	2·576
	∞	2·576	2·576	2·576	2·576	2·576	2·576	2·576	2·576	2·576	2·576
	10	3·581	3·553	3·473	3·350	3·203	3·058	2·939	2·859	2·818	2·807
	12	3·429	3·405	3·338	3·237	3·119	3·003	2·910	2·848	2·816	2·807
	15	3·286	3·267	3·214	3·134	3·042	2·954	2·884	2·838	2·814	2·807
0·5 per cent.	20	3·153	3·139	3·099	3·040	2·974	2·911	2·861	2·829	2·812	2·807
	30	3·030	3·020	2·994	2·955	2·912	2·872	2·841	2·821	2·810	2·807
	60	2·915	2·910	2·897	2·878	2·857	2·838	2·823	2·814	2·809	2·807
	∞	2·807	2·807	2·807	2·807	2·807	2·807	2·807	2·807	2·807	2·807
	10	4·144	4·106	3·999	3·832	3·630	3·425	3·259	3·152	3·103	3·090
	12	3·930	3·898	3·809	3·671	3·508	3·347	3·219	3·138	3·100	3·090
	15	3·733	3·708	3·636	3·528	3·401	3·280	3·185	3·126	3·093	3·090
0·2 per cent.	20	3·552	3·533	3·479	3·399	3·308	3·222	3·156	3·116	3·096	3·090
	30	3·386	3·372	3·336	3·284	3·226	3·172	3·131	3·106	3·094	3·090
	60	3·232	3·225	3·207	3·181	3·153	3·128	3·110	3·098	3·092	3·090
	∞	3·090	3·090	3·090	3·090	3·090	3·090	3·090	3·090	3·090	3·090

TABLE VI. Values of the Correlation Coefficient for Different Levels of Significance

n	·1	·05	·02	·01	·001	n	·1	·05	·02	·01	·001
1	·98769	·99692	·999507	·999877	·9999988	16	·4000	·4683	·5425	·5897	·7084
2	·90000	·95000	·98000	·990000	·99900	17	·3887	·4555	·5285	·5751	·6932
3	·8054	·8783	·93433	·95873	·99116	18	·3783	·4438	·5155	·5614	·6787
4	·7293	·8114	·8822	·91720	·97406	19	·3687	·4329	·5034	·5487	·6652
5	·6694	·7545	·8329	·8745	·95074	20	·3598	·4227	·4921	·5368	·6524
6	·6215	·7067	·7887	·8343	·92493	25	·3233	·3809	·4451	·4869	·5974
7	·5822	·6664	·7498	·7977	·8982	30	·2960	·3494	·4093	·4487	·5541
8	·5494	·6319	·7155	·7646	·8721	35	·2746	·3246	·3810	·4182	·5189
9	·5214	·6021	·6851	·7348	·8471	40	·2573	·3044	·3578	·3932	·4896
10	·4973	·5760	·6581	·7079	·8233	45	·2428	·2875	·3384	·3721	·4648
11	·4762	·5529	·6339	·6835	·8010	50	·2306	·2732	·3218	·3541	·4433
12	·4575	·5324	·6120	·6614	·7800	60	·2108	·2500	·2948	·3248	·4078
13	·4409	·5139	·5923	·6411	·7603	70	·1954	·2319	·2737	·3017	·3799
14	·4259	·4973	·5742	·6226	·7420	80	·1829	·2172	·2565	·2830	·3568
15	·4124	·4821	·5577	·6055	·7246	90	·1726	·2050	·2422	·2673	·3375
						100	·1638	·1946	·2301	·2540	·3211

TABLE VII. Transformation of r to z (Interclass)

z	·00	·01	·02	·03	·04	·05	·06	·07	·08	·09	Mean Diff.
·0	·0000	·0100	·0200	·0300	·0400	·0500	·0599	·0699	·0798	·0898	100
·1	·0997	·1096	·1194	·1293	·1391	·1489	·1586	·1684	·1781	·1877	98
·2	·1974	·2070	·2165	·2260	·2355	·2449	·2543	·2636	·2729	·2821	94
·3	·2913	·3004	·3095	·3185	·3275	·3364	·3452	·3540	·3627	·3714	89
·4	·3800	·3885	·3969	·4053	·4136	·4219	·4301	·4382	·4462	·4542	82
·5	·4621	·4699	·4777	·4854	·4930	·5005	·5080	·5154	·5227	·5299	75
·6	·5370	·5441	·5511	·5580	·5649	·5717	·5784	·5850	·5915	·5980	68
·7	·6044	·6107	·6169	·6231	·6291	·6351	·6411	·6469	·6527	·6584	60
·8	·6640	·6696	·6751	·6805	·6858	·6911	·6963	·7014	·7064	·7114	53
·9	·7163	·7211	·7259	·7306	·7352	·7398	·7443	·7487	·7531	·7574	46
1·0	·7616	·7658	·7699	·7739	·7779	·7818	·7857	·7895	·7932	·7969	39
1·1	·8005	·8041	·8076	·8110	·8144	·8178	·8210	·8243	·8275	·8306	33
1·2	·8337	·8367	·8397	·8426	·8455	·8483	·8511	·8538	·8565	·8591	28
1·3	·8617	·8643	·8668	·8692	·8717	·8741	·8764	·8787	·8810	·8832	24
1·4	·8854	·8875	·8896	·8917	·8937	·8957	·8977	·8996	·9015	·9033	20
1·5	·9051	·9069	·9087	·9104	·9121	·9138	·9154	·9170	·9186	·9201	17
1·6	·9217	·9232	·9246	·9261	·9275	·9289	·9302	·9316	·9329	·9341	14
1·7	·9354	·9366	·9379	·9391	·9402	·9414	·9425	·9436	·9447	·9458	12
1·8	·94681	·94783	·94884	·94983	·95080	·95175	·95268	·95359	·95449	·95537	95
1·9	·95624	·95709	·95792	·95873	·95953	·96032	·96109	·96185	·96259	·96331	79
2·0	·96403	·96473	·96541	·96609	·96675	·96739	·96803	·96865	·96926	·96986	65
2·1	·97045	·97103	·97159	·97215	·97269	·97323	·97375	·97426	·97477	·97526	53
2·2	·97574	·97622	·97668	·97714	·97759	·97803	·97846	·97888	·97929	·97970	44
2·3	·98010	·98049	·98087	·98124	·98161	·98197	·98233	·98267	·98301	·98335	36
2·4	·98367	·98399	·98431	·98462	·98492	·98522	·98551	·98579	·98607	·98635	30
2·5	·98661	·98688	·98714	·98739	·98764	·98788	·98812	·98835	·98858	·98881	24
2·6	·98903	·98924	·98945	·98966	·98987	·99007	·99026	·99045	·99064	·99083	20
2·7	·99101	·99118	·99136	·99153	·99170	·99186	·99202	·99218	·99233	·99248	16
2·8	·99263	·99278	·99292	·99306	·99320	·99333	·99346	·99359	·99372	·99384	13
2·9	·99396	·99408	·99420	·99431	·99443	·99454	·99464	·99475	·99485	·99495	11
	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	
3	·99505	·99595	·99668	·99728	·99777	·99818	·99851	·99878	·99900	·99918	—
4	·99933	·99945	·99955	·99963	·99970	·99975	·99980	·99983	·99986	·99989	—

For notes see foot of opposite page.

TABLE VIII. TESTS OF SIGNIFICANCE FOR 2×2 CONTINGENCY TABLES

Calculate χ_c, the square root of χ^2 corrected for continuity (Fisher, *Statistical Methods*, 21.01). Determine m = the smallest expectation of any class, and $p = \dfrac{\text{the smallest expectation}}{\text{the smallest marginal total}}$.

The table gives the 2·5 and 0·5 per cent. points of χ_c for each tail separately, these being different for the two tails except when $p = 0{\cdot}5$. The observed set of values will lie on the longer tail if the observed number in the cell with smallest expectation is greater than expectation.

Since three of the four marginal totals are independent their values are not completely determined by m and p. For a given m and p there is a whole set of contingency distributions, and a corresponding set of values of χ_c for each level of significance. The variation of these χ_c is, however, small. The table gives the greatest and least values, *i.e.* those for the binomial distribution (Roman type), and the limiting contingency distribution (black type). When $p = 0$ there is only one value, this being derived from the Poisson distribution.

The table should always be used in place of the ordinary χ^2 table when m is less than 10. Even with values of m as great as 100 the gain in precision is appreciable when p is small and the interest centres in the probability of a single tail (*i.e.* deviations in one direction only). The correction for continuity should be used when m is less than 500.

In regions of the table where no values are given, and to determine the exact probability in doubtful cases, the exact solution (on which this table is based) must be used (Fisher, *Statistical Methods*, 21.02).

The table can be used for testing the deviation of a sample of a binomial distribution with known p. Remember to include the contributions to χ^2 from both classes when calculating χ_c.

	p \ m	1	2	3	4	5	6	8	12	24	48	96
P = ·025 . .	0	...	...	...	1·68	1·71	1·74	1·77	1·81	1·86	1·89	1·91
Shorter tail . .	0·25	...	...	1·73	1·77	1·80	1·82	1·84	1·87	1·90	1·91	1·93
		...	...	1·83	1·85	1·86	1·87	1·89	1·90	1·92	1·94	1·94
	0·5	...	...	1·88	1·90	1·91	1·92	1·93	1·94	1·95	1·96	1·96
		...	1·91	1·93	1·94	1·94	1·94	1·95	1·95	1·95	1·96	1·96
Longer tail . .	0·25	2·10	2·08	2·07	2·06	2·06	2·05	2·04	2·02	2·01	2·00	1·99
		2·04	2·03	2·02	2·02	2·01	2·01	2·00	1·99	1·99	1·98	1·97
	0	2·32	2·24	2·19	2·16	2·14	2·13	2·11	2·08	2·05	2·03	2·01
P = ·005 . .	0	...	...	...	...	2·06	2·13	2·19	2·27	2·37	2·43	2·48
Shorter tail . .	0·25	...	...	...	2·18	2·23	2·27	2·32	2·38	2·45	2·49	2·52
		...	...	...	2·32	2·36	2·38	2·42	2·46	2·50	2·52	2·54
	0·5	...	...	...	2·41	2·44	2·47	2·50	2·52	2·55	2·56	2·57
		...	...	2·48	2·50	2·52	2·53	2·54	2·55	2·56	2·57	2·57
Longer tail . .	0·25	2·79	2·79	2·78	2·76	2·75	2·73	2·72	2·70	2·67	2·64	2·63
		2·67	2·67	2·67	2·67	2·66	2·66	2·65	2·64	2·63	2·62	2·60
	0	3·33	3·13	3·05	2·97	2·95	2·92	2·88	2·83	2·76	2·70	2·67

Notes on Tables VI and VII.

For a total correlation, n is 2 less than the number of pairs in the sample; for a partial correlation, the number of eliminated variates also should be subtracted. The probability given at the head of each column of Table VI represents the chance that r will be greater than the values given or less than *minus* these values.

Table VII gives the transformation $r = (e^{2z}-1)/(e^{2z}+1)$ or $z = \frac{1}{2}\{\log_e(1+r) - \log_e(1-r)\}$. With n defined as above z is distributed approximately normally with variance $1/(n-1)$. For exact work correct for bias in z by subtracting $r/2(n+1)$ from z.

The table is also sometimes useful for transforming the probability scale, being equivalent to $z = \frac{1}{2}\{\log p - \log(1-p)\}$ where $r = 2p-1$. (See Introduction).

TABLE VIII1. BINOMIAL AND POISSON DISTRIBUTIONS: LIMITS OF THE EXPECTATION

(Based on W. L. Stevens)

a	N	$p = a/N$	Probability P of a or more ·005	·025	·1	Probability P of a or fewer ·1	·025	·005
0	5	(·2)	For $a = 0$, $1-P^{1/N}$			1·84	2·61	3·27
		(·15)	is the greatest prob-			1·95	2·83	3·66
	10	(·1)	ability allowable.			2·06	3·09	4·11
	20	(·05)				2·17	3·37	4·65
	∞	0				2·30	3·69	5·30
1	2		·0050	·0252	·103	1·897	1·9748	1·9950
	3		·0050	·0252	·104	2·413	2·7171	2·8758
	4		·0050	·0252	·104	2·718	3·224	3·556
	5	·2	·0050	·0253	·104	2·92	3·58	4·07
		·15	·0050	·0253	·105	3·14	3·99	4·69
	10	·1	·0050	·0253	·105	3·37	4·45	5·44
	20	·05	·0050	·0253	·105	3·62	4·97	6·34
	∞	0	·0050	·0253	·105	3·89	5·57	7·43
2	4		·118	·270	·570	3·430	3·730	3·882
	5		·114	·264	·561	3·77	4·267	4·586
	6		·112	·260	·556	4·00	4·67	5·138
	7		·111	·257	·552	4·17	4·97	5·54
	8	·25	·110	·255	·549	4·31	5·21	5·94
	10	·2	·108	·252	·545	4·50	5·56	6·48
		·15	·107	·249	·542	4·69	5·94	7·09
	20	·1	·106	·247	·538	4·90	6·34	7·74
	40	·05	·105	·245	·535	5·11	6·77	8·44
	∞	0	·103	·242	·532	5·32	7·22	9·27
3	6		·398	·709	1·21	4·79	5·291	5·602
	7		·387	·693	1·19	5·05	5·71	6·181
	8		·380	·682	1·17	5·24	6·04	6·642
	9		·374	·674	1·17	5·39	6·31	7·03
	10	·3	·370	·667	1·16	5·52	6·52	7·35
	15	·2	·358	·650	1·14	5·89	7·21	8·41
	30	·1	·348	·634	1·12	6·28	7·96	9·61
	∞	0	·338	·619	1·10	6·68	8·77	10·98
4	8	·5	·799	1·26	1·92	6·08	6·74	7·201
	10	·4	·768	1·22	1·88	6·46	7·38	8·091
		·3	·741	1·18	1·84	6·83	8·04	9·06
	20	·2	·716	1·15	1·80	7·21	8·73	10·13
	40	·1	·694	1·12	1·77	7·60	9·47	11·31
	∞	0	·672	1·09	1·74	7·99	10·24	12·59
5	10	·5	1·28	1·87	2·67	7·33	8·13	8·72
		·4	1·23	1·81	2·61	7·72	8·79	9·66
		·3	1·19	1·76	2·56	8·10	9·47	10·67
	25	·2	1·15	1·71	2·52	8·49	10·18	11·75
	50	·1	1·11	1·66	2·47	8·88	10·91	12·90
	∞	0	1·08	1·62	2·43	9·27	11·67	14·15
6	12	·5	1·83	2·53	3·46	8·54	9·47	10·17
	15	·4	1·75	2·45	3·38	8·95	10·16	11·16
	20	·3	1·69	2·38	3·32	9·35	10·86	12·19
	30	·2	1·63	2·31	3·26	9·74	11·57	13·28
	60	·1	1·58	2·26	3·20	10·14	12·30	14·44
	∞	0	1·54	2·20	3·15	10·53	13·06	15·66
7	14	·5	2·41	3·23	4·26	9·74	10·77	11·59
		·4	2·32	3·12	4·17	10·15	11·48	12·61
		·3	2·24	3·03	4·09	10·56	12·20	13·67
	35	·2	2·16	2·95	4·02	10·97	12·93	14·77
	70	·1	2·10	2·88	3·96	11·37	13·67	15·92
	∞	0	2·04	2·81	3·89	11·77	14·42	17·13

a	$p = a/N$	Probability P of a or more ·005	·025	·1	Probability P of a or fewer ·1	·025	·005
8	·5	3·04	3·94	5·09	10·91	12·06	12·96
	·4	2·92	3·82	4·98	11·35	12·79	14·02
	·3	2·82	3·72	4·89	11·77	13·52	15·10
	·2	2·73	3·62	4·80	12·18	14·26	16·22
	·1	2·65	3·53	4·73	12·59	15·01	17·37
	0	2·57	3·45	4·66	12·99	15·76	18·58
9	·5	3·68	4·68	5·92	12·08	13·32	14·32
	·4	3·55	4·54	5·80	12·52	14·07	15·40
	·3	3·43	4·42	5·70	12·96	14·82	16·50
	·2	3·32	4·31	5·60	13·38	15·57	17·63
	·1	3·22	4·21	5·51	13·79	16·32	18·80
	0	3·13	4·12	5·43	14·21	17·08	20·00
10	·5	4·35	5·44	6·76	13·24	14·56	15·65
	·4	4·20	5·28	6·63	13·69	15·33	16·76
	·3	4·06	5·14	6·51	14·13	16·10	17·88
	·2	3·93	5·02	6·41	14·56	16·86	19·02
	·1	3·82	4·90	6·31	14·99	17·62	20·20
	0	3·72	4·80	6·22	15·41	18·39	21·40
9	·5	+0·14	−0·16	−0·36	+0·36	+0·16	−0·14
	·4	+0·53	+0·10	−0·22	+0·55	+0·52	+0·41
	·3	+0·89	+0·34	−0·09	+0·74	+0·90	+1·04
	·2	+1·23	+0·57	+0·04	+0·94	+1·31	+1·72
	·1	+1·55	+0·79	+0·16	+1·15	+1·74	+2·47
	0	+1·86	+1·00	+0·28	+1·36	+2·20	+3·27
16	·5	−0·02	−0·25	−0·40	+0·40	+0·25	+0·02
	·4	+0·40	+0·02	−0·26	+0·58	+0·60	+0·57
	·3	+0·79	+0·28	−0·12	+0·76	+0·96	+1·16
	·2	+1·17	+0·52	+0·01	+0·94	+1·34	+1·80
	·1	+1·52	+0·76	+0·14	+1·13	+1·73	+2·47
	0	+1·87	+0·99	+0·26	+1·33	+2·14	+3·18
36	·5	−0·18	−0·34	−0·44	+0·44	+0·34	+0·18
	·4	+0·26	−0·06	−0·29	+0·60	+0·67	+0·72
	·3	+0·68	+0·21	−0·15	+0·77	+1·01	+1·28
	·2	+1·09	+0·47	−0·02	+0·94	+1·36	+1·86
	·1	+1·49	+0·73	+0·12	+1·11	+1·71	+2·46
	0	+1·88	+0·97	+0·25	+1·29	+2·08	+3·08
144	·5	−0·35	−0·43	−0·47	+0·47	+0·43	+0·35
	·4	+0·12	−0·14	−0·33	+0·63	+0·73	+0·86
	·3	+0·57	+0·14	−0·18	+0·78	+1·05	+1·37
	·2	+1·01	+0·42	−0·04	+0·94	+1·37	+1·90
	·1	+1·45	+0·69	+0·09	+1·09	+1·69	+2·44
	0	+1·88	+0·96	+0·23	+1·25	+2·01	+2·98
∞	·5	−0·50	−0·50	−0·50	+0·50	+0·50	+0·50
	·4	−0·02	−0·21	−0·36	+0·64	+0·79	+0·98
	·3	+0·45	+0·08	−0·21	+0·79	+1·08	+1·45
	·2	+0·93	+0·37	−0·07	+0·93	+1·37	+1·93
	·1	+1·40	+0·66	+0·07	+1·07	+1·66	+2·40
	0	+1·88	+0·95	+0·21	+1·21	+1·95	+2·88

To obtain the limits of the probability of an event (observed to occur a times out of N) corresponding to a given probability level P, divide the tabulated values by N, first interpolating if necessary by linear interpolation with reference to $p = a/N$. The limits of the expectation of Poisson distributions are given directly, taking $p = 0$. For $a > \frac{1}{2}N$ enter the table with $a' = N - a$. For $a > 10$ add the tabulated corrections to the limits of the expectation calculated from the standard error $\sqrt{\{a(1-p)\}}$, using asymptotic interpolation on $12/\sqrt{a}$.

TABLE VIII2. DENSITIES OF ORGANISMS ESTIMATED BY THE DILUTION METHOD

(W. L. Stevens)

Two-fold

x	Number of Levels (s). 4	5	6	7	8	9	10	11 or more
0·4	·757	·773	·781	·785	·787	·788	·789	·789
0·6	·622	·640	·649	·653	·655	·656	·657	·657
0·8	·537	·556	·566	·571	·573	·574	·575	·575
1·0	·479	·500	·511	·516	·518	·520	·520	·521
1·2	·437	·461	·472	·478	·480	·482	·482	·483
1·4	·406	·432	·444	·450	·453	·455	·456	·456
1·6	·381	·411	·424	·431	·435	·436	·437	·438
1·8	·361	·394	·410	·417	·421	·423	·424	·425
2·0	·344	·382	·399	·408	·412	·414	·415	·416
2·5		·358	·382	·394	·399	·402	·403	·405
3·0			·370	·386	·394	·398	·400	·402
3·5				·379	·390	·396	·399	·401
4·0					·386	·394	·397	·401
4·5						·390	·396	·401
5·0							·394	·401
								·401*
y								
7·0								·399
6·0								·397
5·0							·394	·394
4·5						·390	·390	·390
4·0					·386	·386	·386	·386
3·5				·379	·379	·379	·379	·379
3·0			·370	·370	·370	·370	·370	·370
2·5		·358	·356	·356	·356	·356	·356	·356
2·0	·344	·334	·334	·334	·334	·334	·334	·334
1·8	·327	·323	·323	·323	·323	·323	·323	·323
1·6	·311	·309	·309	·309	·309	·309	·309	·309
1·4	·293	·292	·292	·292	·292	·292	·292	·292
1·2	·271	·271	·271	·271	·271	·271	·271	·271
1·0	·245	·245	·245	·245	·245	·245	·245	·245
0·8	·212	·212	·212	·212	·212	·212	·212	·212
0·6	·167	·167	·167	·167	·167	·167	·167	·167
0·4	·101	·101	·101	·101	·101	·101	·101	·101

Four-fold

x	Number of Levels 4	5	6 or more
0·4	·704	·706	·707
0·6	·615	·617	·618
0·8	·573	·576	·577
1·0	·555	·558	·559
1·5	·545	·551	·553
2·0	·537	·548	·551
2·5		·545	·552
			·552*
y			
3·5			·550
3·0			·548
2·5		·545	·545
2·0	·537	·537	·537
1·5	·522	·522	·522
1·0	·488	·488	·488
0·8	·464	·464	·464
0·6	·431	·431	·431
0·4	·375	·375	·375

Ten-fold (Three or more levels)

$x<1$		$x>1, y>2$			$y\leqslant 2$			
x	K	x	K	y	K	y	K	y
		·0	·763	·0	·761	2·0	·744	1·0
		·1	·768	·9	·766	1·9	·744	0·9
		·2	·768	·8	·764	1·8	·734	0·8
		·3	·760	·7	·755	1·7	·712	0·7
0·4	·761	·4	·747	·6	·741	1·6	·684	0·6
0·5	·740	·5	·736	·5	·729	1·5	·658	0·5
0·6	·733	·6	·733	·4	·724	1·4	·638	0·4
0·7	·736	·7	·736	·3	·726	1·3		
0·8	·744	·8	·744	·2	·732	1·2		
0·9	·753	·9	·753	·1	·739	1·1		
1·0	·763	·0	·763	·0	·744	1·0		

When $x>1$ and $y>2$ enter the table with the decimal part of x or y only.

Calculate the mean fertile level x, and the mean sterile level y, where x is the number of fertile plates/number of cultures at each level (n), and $x+y$ is the number of levels. Enter the table with x or y, as indicated, and determine the corresponding value of the tabular entry K. When x and y fall outside the tabulated range use the value marked with an asterisk (*). The estimate of the number, λ, of organisms in the quantity of the medium used for one culture at the highest concentration is then given by $\log \lambda = x \log a - K$, where a is the dilution factor. The average value of the variance of the mean fertile level is $\frac{1}{n}\frac{\log 2}{\log a}$, and the average value of the variance of $\log \lambda$ is $\frac{1}{y}\log 2 \log a$ (see Introduction). Thus:

Two-fold

$$\log \lambda = 0{\cdot}30103\, x - K$$
$$\overline{V(x)} = 1/n$$
$$\overline{V(\log \lambda)} = 0{\cdot}091/n$$

Four-fold.

$$\log \lambda = 0{\cdot}60206\, x - K$$
$$\overline{V(x)} = 1/2n$$
$$\overline{V(\log \lambda)} = 0{\cdot}201/n$$

Ten-fold.

$$\log \lambda = x - K$$
$$\overline{V(x)} = 0{\cdot}301/n$$

Table IX. Probits

Transformation of the Sigmoid Dosage Mortality Curve to a Straight Line. (C. I. Bliss.)

	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1	2	3	4	5
0	...	1·9098	2·1218	2·2522	2·3479	2·4242	2·4879	2·5427	2·5911	2·6344					
1	2·6737	2·7096	2·7429	2·7738	2·8027	2·8299	2·8556	2·8799	2·9031	2·9251					
2	2·9463	2·9665	2·9859	3·0046	3·0226	3·0400	3·0569	3·0732	3·0890	3·1043	For more detail see values for 95-100.				
3	3·1192	3·1337	3·1478	3·1616	3·1750	3·1881	3·2009	3·2134	3·2256	3·2376					
4	3·2493	3·2608	3·2721	3·2831	3·2940	3·3046	3·3151	3·3253	3·3354	3·3454					
5	3·3551	3·3648	3·3742	3·3836	3·3928	3·4018	3·4107	3·4195	3·4282	3·4368	9	18	27	36	45
6	3·4452	3·4536	3·4618	3·4699	3·4780	3·4859	3·4937	3·5015	3·5091	3·5167	8	16	24	32	40
7	3·5242	3·5316	3·5389	3·5462	3·5534	3·5605	3·5675	3·5745	3·5813	3·5882	7	14	21	28	36
8	3·5949	3·6016	3·6083	3·6148	3·6213	3·6278	3·6342	3·6405	3·6468	3·6531	6	13	19	26	32
9	3·6592	3·6654	3·6715	3·6775	3·6835	3·6894	3·6953	3·7012	3·7070	3·7127	6	12	18	24	30
10	3·7184	3·7241	3·7298	3·7354	3·7409	3·7464	3·7519	3·7574	3·7628	3·7681	6	11	17	22	28
11	3·7735	3·7788	3·7840	3·7893	3·7945	3·7996	3·8048	3·8099	3·8150	3·8200	5	10	16	21	26
12	3·8250	3·8300	3·8350	3·8399	3·8448	3·8497	3·8545	3·8593	3·8641	3·8689	5	10	15	20	24
13	3·8736	3·8783	3·8830	3·8877	3·8923	3·8969	3·9015	3·9061	3·9107	3·9152	5	9	14	18	23
14	3·9197	3·9242	3·9286	3·9331	3·9375	3·9419	3·9463	3·9506	3·9550	3·9593	4	9	13	18	22
15	3·9636	3·9678	3·9721	3·9763	3·9806	3·9848	3·9890	3·9931	3·9973	4·0014	4	8	13	17	21
16	4·0055	4·0096	4·0137	4·0178	4·0218	4·0259	4·0299	4·0339	4·0379	4·0419	4	8	12	16	20
17	4·0458	4·0498	4·0537	4·0576	4·0615	4·0654	4·0693	4·0731	4·0770	4·0808	4	8	12	16	19
18	4·0846	4·0884	4·0922	4·0960	4·0998	4·1035	4·1073	4·1110	4·1147	4·1184	4	8	11	15	19
19	4·1221	4·1258	4·1295	4·1331	4·1367	4·1404	4·1440	4·1476	4·1512	4·1548	4	7	11	15	18
20	4·1584	4·1619	4·1655	4·1690	4·1726	4·1761	4·1796	4·1831	4·1866	4·1901	4	7	11	14	18
21	4·1936	4·1970	4·2005	4·2039	4·2074	4·2108	4·2142	4·2176	4·2210	4·2244	3	7	10	14	17
22	4·2278	4·2312	4·2345	4·2379	4·2412	4·2446	4·2479	4·2512	4·2546	4·2579	3	7	10	13	17
23	4·2612	4·2644	4·2677	4·2710	4·2743	4·2775	4·2808	4·2840	4·2872	4·2905	3	7	10	13	16
24	4·2937	4·2969	4·3001	4·3033	4·3065	4·3097	4·3129	4·3160	4·3192	4·3224	3	6	10	13	16
25	4·3255	4·3287	4·3318	4·3349	4·3380	4·3412	4·3443	4·3474	4·3505	4·3536	3	6	9	12	16
26	4·3567	4·3597	4·3628	4·3659	4·3689	4·3720	4·3750	4·3781	4·3811	4·3842	3	6	9	12	15
27	4·3872	4·3902	4·3932	4·3962	4·3992	4·4022	4·4052	4·4082	4·4112	4·4142	3	6	9	12	15
28	4·4172	4·4201	4·4231	4·4260	4·4290	4·4319	4·4349	4·4378	4·4408	4·4437	3	6	9	12	15
29	4·4466	4·4495	4·4524	4·4554	4·4583	4·4612	4·4641	4·4670	4·4698	4·4727	3	6	9	12	14
30	4·4756	4·4785	4·4813	4·4842	4·4871	4·4899	4·4928	4·4956	4·4985	4·5013	3	6	9	11	14
31	4·5041	4·5070	4·5098	4·5126	4·5155	4·5183	4·5211	4·5239	4·5267	4·5295	3	6	8	11	14
32	4·5323	4·5351	4·5379	4·5407	4·5435	4·5462	4·5490	4·5518	4·5546	4·5573	3	6	8	11	14
33	4·5601	4·5628	4·5656	4·5684	4·5711	4·5739	4·5766	4·5793	4·5821	4·5848	3	5	8	11	14
34	4·5875	4·5903	4·5930	4·5957	4·5984	4·6011	4·6039	4·6066	4·6093	4·6120	3	5	8	11	14
35	4·6147	4·6174	4·6201	4·6228	4·6255	4·6281	4·6308	4·6335	4·6362	4·6389	3	5	8	11	13
36	4·6415	4·6442	4·6469	4·6495	4·6522	4·6549	4·6575	4·6602	4·6628	4·6655	3	5	8	11	13
37	4·6681	4·6708	4·6734	4·6761	4·6787	4·6814	4·6840	4·6866	4·6893	4·6919	3	5	8	11	13
38	4·6945	4·6971	4·6998	4·7024	4·7050	4·7076	4·7102	4·7129	4·7155	4·7181	3	5	8	10	13
39	4·7207	4·7233	4·7259	4·7285	4·7311	4·7337	4·7363	4·7389	4·7415	4·7441	3	5	8	10	13
40	4·7467	4·7492	4·7518	4·7544	4·7570	4·7596	4·7622	4·7647	4·7673	4·7699	3	5	8	10	13
41	4·7725	4·7750	4·7776	4·7802	4·7827	4·7853	4·7879	4·7904	4·7930	4·7955	3	5	8	10	13
42	4·7981	4·8007	4·8032	4·8058	4·8083	4·8109	4·8134	4·8160	4·8185	4·8211	3	5	8	10	13
43	4·8236	4·8262	4·8287	4·8313	4·8338	4·8363	4·8389	4·8414	4·8440	4·8465	3	5	8	10	13
44	4·8490	4·8516	4·8541	4·8566	4·8592	4·8617	4·8642	4·8668	4·8693	4·8718	3	5	8	10	13
45	4·8743	4·8769	4·8794	4·8819	4·8844	4·8870	4·8895	4·8920	4·8945	4·8970	3	5	8	10	13
46	4·8996	4·9021	4·9046	4·9071	4·9096	4·9122	4·9147	4·9172	4·9197	4·9222	3	5	8	10	13
47	4·9247	4·9272	4·9298	4·9323	4·9348	4·9373	4·9398	4·9423	4·9448	4·9473	3	5	8	10	13
48	4·9498	4·9524	4·9549	4·9574	4·9599	4·9624	4·9649	4·9674	4·9699	4·9724	3	5	8	10	13
49	4·9749	4·9774	4·9799	4·9825	4·9850	4·9875	4·9900	4·9925	4·9950	4·9975	3	5	8	10	13

The probit corresponding to a given percentage is the normal deviate (increased by 5 to avoid negative values) exceeding this percentage in the population (see Table I).

TABLE IX. PROBITS—*continued*

	0·0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1	2	3	4	5
50	5·0000	5·0025	5·0050	5·0075	5·0100	5·0125	5·0150	5·0175	5·0201	5·0226	3	5	8	10	13
51	5·0251	5·0276	5·0301	5·0326	5·0351	5·0376	5·0401	5·0426	5·0451	5·0476	3	5	8	10	13
52	5·0502	5·0527	5·0552	5·0577	5·0602	5·0627	5·0652	5·0677	5·0702	5·0728	3	5	8	10	13
53	5·0753	5·0778	5·0803	5·0828	5·0853	5·0878	5·0904	5·0929	5·0954	5·0979	3	5	8	10	13
54	5·1004	5·1030	5·1055	5·1080	5·1105	5·1130	5·1156	5·1181	5·1206	5·1231	3	5	8	10	13
55	5·1257	5·1282	5·1307	5·1332	5·1358	5·1383	5·1408	5·1434	5·1459	5·1484	3	5	8	10	13
56	5·1510	5·1535	5·1560	5·1586	5·1611	5·1637	5·1662	5·1687	5·1713	5·1738	3	5	8	10	13
57	5·1764	5·1789	5·1815	5·1840	5·1866	5·1891	5·1917	5·1942	5·1968	5·1993	3	5	8	10	13
58	5·2019	5·2045	5·2070	5·2096	5·2121	5·2147	5·2173	5·2198	5·2224	5·2250	3	5	8	10	13
59	5·2275	5·2301	5·2327	5·2353	5·2378	5·2404	5·2430	5·2456	5·2482	5·2508	3	5	8	10	13
60	5·2533	5·2559	5·2585	5·2611	5·2637	5·2663	5·2689	5·2715	5·2741	5·2767	3	5	8	10	13
61	5·2793	5·2819	5·2845	5·2871	5·2898	5·2924	5·2950	5·2976	5·3002	5·3029	3	5	8	10	13
62	5·3055	5·3081	5·3107	5·3134	5·3160	5·3186	5·3213	5·3239	5·3266	5·3292	3	5	8	11	13
63	5·3319	5·3345	5·3372	5·3398	5·3425	5·3451	5·3478	5·3505	5·3531	5·3558	3	5	8	11	13
64	5·3585	5·3611	5·3638	5·3665	5·3692	5·3719	5·3745	5·3772	5·3799	5·3826	3	5	8	11	13
65	5·3853	5·3880	5·3907	5·3934	5·3961	5·3989	5·4016	5·4043	5·4070	5·4097	3	5	8	11	14
66	5·4125	5·4152	5·4179	5·4207	5·4234	5·4261	5·4289	5·4316	5·4344	5·4372	3	5	8	11	14
67	5·4399	5·4427	5·4454	5·4482	5·4510	5·4538	5·4565	5·4593	5·4621	5·4649	3	6	8	11	14
68	5·4677	5·4705	5·4733	5·4761	5·4789	5·4817	5·4845	5·4874	5·4902	5·4930	3	6	8	11	14
69	5·4959	5·4987	5·5015	5·5044	5·5072	5·5101	5·5129	5·5158	5·5187	5·5215	3	6	9	11	14
70	5·5244	5·5273	5·5302	5·5330	5·5359	5·5388	5·5417	5·5446	5·5476	5·5505	3	6	9	12	14
71	5·5534	5·5563	5·5592	5·5622	5·5651	5·5681	5·5710	5·5740	5·5769	5·5799	3	6	9	12	15
72	5·5828	5·5858	5·5888	5·5918	5·5948	5·5978	5·6008	5·6038	5·6068	5·6098	3	6	9	12	15
73	5·6128	5·6158	5·6189	5·6219	5·6250	5·6280	5·6311	5·6341	5·6372	5·6403	3	6	9	12	15
74	5·6433	5·6464	5·6495	5·6526	5·6557	5·6588	5·6620	5·6651	5·6682	5·6713	3	6	9	12	16
75	5·6745	5·6776	5·6808	5·6840	5·6871	5·6903	5·6935	5·6967	5·6999	5·7031	3	6	10	13	16
76	5·7063	5·7095	5·7128	5·7160	5·7192	5·7225	5·7257	5·7290	5·7323	5·7356	3	7	10	13	16
77	5·7388	5·7421	5·7454	5·7488	5·7521	5·7554	5·7588	5·7621	5·7655	5·7688	3	7	10	13	17
78	5·7722	5·7756	5·7790	5·7824	5·7858	5·7892	5·7926	5·7961	5·7995	5·8030	3	7	10	14	17
79	5·8064	5·8099	5·8134	5·8169	5·8204	5·8239	5·8274	5·8310	5·8345	5·8381	4	7	11	14	18
80	5·8416	5·8452	5·8488	5·8524	5·8560	5·8596	5·8633	5·8669	5·8705	5·8742	4	7	11	14	18
81	5·8779	5·8816	5·8853	5·8890	5·8927	5·8965	5·9002	5·9040	5·9078	5·9116	4	7	11	15	19
82	5·9154	5·9192	5·9230	5·9269	5·9307	5·9346	5·9385	5·9424	5·9463	5·9502	4	8	12	15	19
83	5·9542	5·9581	5·9621	5·9661	5·9701	5·9741	5·9782	5·9822	5·9863	5·9904	4	8	12	16	20
84	5·9945	5·9986	6·0027	6·0069	6·0110	6·0152	6·0194	6·0237	6·0279	6·0322	4	8	13	17	21
85	6·0364	6·0407	6·0450	6·0494	6·0537	6·0581	6·0625	6·0669	6·0714	6·0758	4	9	13	18	22
86	6·0803	6·0848	6·0893	6·0939	6·0985	6·1031	6·1077	6·1123	6·1170	6·1217	5	9	14	18	23
87	6·1264	6·1311	6·1359	6·1407	6·1455	6·1503	6·1552	6·1601	6·1650	6·1700	5	10	15	19	24
88	6·1750	6·1800	6·1850	6·1901	6·1952	6·2004	6·2055	6·2107	6·2160	6·2212	5	10	15	21	26
89	6·2265	6·2319	6·2372	6·2426	6·2481	6·2536	6·2591	6·2646	6·2702	6·2759	5	11	16	22	27
90	6·2816	6·2873	6·2930	6·2988	6·3047	6·3106	6·3165	6·3225	6·3285	6·3346	6	12	18	24	29
91	6·3408	6·3469	6·3532	6·3595	6·3658	6·3722	6·3787	6·3852	6·3917	6·3984	6	13	19	26	32
92	6·4051	6·4118	6·4187	6·4255	6·4325	6·4395	6·4466	6·4538	6·4611	6·4684	7	14	21	28	35
93	6·4758	6·4833	6·4909	6·4985	6·5063	6·5141	6·5220	6·5301	6·5382	6·5464	8	16	24	31	39
94	6·5548	6·5632	6·5718	6·5805	6·5893	6·5982	6·6072	6·6164	6·6258	6·6352	9	18	27	36	45
95	6·6449	6·6546	6·6646	6·6747	6·6849	6·6954	6·7060	6·7169	6·7279	6·7392					
	97	100	101	102	105	106	109	110	113	115					
96	6·7507	6·7624	6·7744	6·7866	6·7991	6·8119	6·8250	6·8384	6·8522	6·8663					
	117	120	122	125	128	131	134	138	141	145					
97	6·8808	6·8957	6·9110	6·9268	6·9431	6·9600	6·9774	6·9954	7·0141	7·0335					
	149	153	158	163	169	174	180	187	194	202					

Continued on next page.

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Table IX. Probits—*continued*

	0·00	0·01	0·02	0·03	0·04	0·05	0·06	0·07	0·08	0·09	1	2	3	4	5
98·0	7·0537	7·0558	7·0579	7·0600	7·0621	7·0642	7·0663	7·0684	7·0706	7·0727	2	4	6	8	11
98·1	7·0749	7·0770	7·0792	7·0814	7·0836	7·0858	7·0880	7·0902	7·0924	7·0947	2	4	7	9	11
98·2	7·0969	7·0992	7·1015	7·1038	7·1061	7·1084	7·1107	7·1130	7·1154	7·1177	2	5	7	9	12
98·3	7·1201	7·1224	7·1248	7·1272	7·1297	7·1321	7·1345	7·1370	7·1394	7·1419	2	5	7	10	12
98·4	7·1444	7·1469	7·1494	7·1520	7·1545	7·1571	7·1596	7·1622	7·1648	7·1675	3	5	8	10	13
98·5	7·1701	7·1727	7·1754	7·1781	7·1808	7·1835	7·1862	7·1890	7·1917	7·1945	3	5	8	11	14
98·6	7·1973	7·2001	7·2029	7·2058	7·2086	7·2115	7·2144	7·2173	7·2203	7·2232	3	6	9	12	14
98·7	7·2262	7·2292	7·2322	7·2353	7·2383	7·2414	7·2445	7·2476	7·2508	7·2539	3	6	9	12	15
98·8	7·2571	7·2603	7·2636	7·2668	7·2701	7·2734	7·2768	7·2801	7·2835	7·2869	3	7	10	13	17
98·9	7·2904	7·2938	7·2973	7·3009	7·3044	7·3080	7·3116	7·3152	7·3189	7·3226	4	7	11	14	18
99·0	7·3263	7·3301	7·3339	7·3378	7·3416	7·3455	7·3495	7·3535	7·3575	7·3615	4	8	12	16	20
99·1	7·3656	7·3698	7·3739	7·3781	7·3824	7·3867	7·3911	7·3954	7·3999	7·4044	4	9	13	17	22
99·2	7·4089	7·4135	7·4181	7·4228	7·4276	7·4324	7·4372	7·4422	7·4471	7·4522	5	10	14	19	24
99·3	7·4573	7·4624	7·4677	7·4730	7·4783	7·4838	7·4893	7·4949	7·5006	7·5063	5	11	16	22	27
99·4	7·5121	7·5181	7·5241	7·5302	7·5364	7·5427	7·5491	7·5556	7·5622	7·5690	6	13	19	25	32
99·5	7·5758	7·5828	7·5899	7·5972	7·6045	7·6121	7·6197	7·6276	7·6356	7·6437					
99·6	7·6521	7·6606	7·6693	7·6783	7·6874	7·6968	7·7065	7·7164	7·7266	7·7370					
99·7	7·7478	7·7589	7·7703	7·7822	7·7944	7·8070	7·8202	7·8338	7·8480	7·8627					
99·8	7·8782	7·8943	7·9112	7·9290	7·9478	7·9677	7·9889	8·0115	8·0357	8·0618					
99·9	8·0902	8·1214	8·1559	8·1947	8·2389	8·2905	8·3528	8·4316	8·5401	8·7190					

Table X. Simple Quantiles of the Normal Distribution

Proper Fractions, with Denominators up to 30, and the corresponding Normal Deviates

*1/30	1·8339	*1/10	1·2816	5/27	0·8958	8/29	0·5952	*11/30	0·3407	9/20	0·1257
1/29	1·8187	3/29	1·2621	3/16	0·8871	5/18	0·5895	7/19	0·3360	5/11	0·1142
1/28	1·8028	2/19	1·2521	4/21	0·8761	7/25	0·5828	10/27	0·3309	11/24	0·1046
1/27	1·7861	3/28	1·2419	5/26	0·8694	2/7	0·5661	3/8	0·3186	6/13	0·0966
1/26	1·7688	1/9	1·2206	*1/5	0·8416	7/24	0·5485	11/29	0·3073	13/28	0·0896
1/25	1·7507	3/26	1·1984	6/29	0·8172	5/17	0·5414	8/21	0·3030	*7/15	0·0836
1/24	1·7317	2/17	1·1868	5/24	0·8122	8/27	0·5351	5/13	0·2934	8/17	0·0738
1/23	1·7117	3/25	1·1750	4/19	0·8046	*3/10	0·5244	7/18	0·2822	9/19	0·0660
1/22	1·6906	1/8	1·1503	3/14	0·7916	7/23	0·5119	9/23	0·2759	10/21	0·0597
1/21	1·6684	3/23	1·1244	5/23	0·7810	4/13	0·5024	11/28	0·2719	11/23	0·0545
1/20	1·6449	*2/15	1·1108	2/9	0·7647	9/29	0·4949	*2/5	0·2533	12/25	0·0502
1/19	1·6199	3/22	1·0968	5/22	0·7479	5/16	0·4888	11/27	0·2342	13/27	0·0464
1/18	1·5932	4/29	1·0897	3/13	0·7363	6/19	0·4795	9/22	0·2299	14/29	0·0432
1/17	1·5648	1/7	1·0676	*7/30	0·7279	7/22	0·4728	7/17	0·2230	*1/2	0
1/16	1·5341	4/27	1·0444	4/17	0·7215	8/25	0·4677	12/29	0·2178		
*1/15	1·5011	3/20	1·0364	5/21	0·7124	9/28	0·4637	5/12	0·2104		
2/29	1·4835	2/13	1·0201	6/25	0·7063	*1/3	0·4307	8/19	0·1992		
1/14	1·4652	3/19	1·0031	7/29	0·7019	10/29	0·3993	11/26	0·1940		
2/27	1·4461	4/25	0·9945	1/4	0·6745	9/26	0·3957	3/7	0·1800		
1/13	1·4261	*1/6	0·9674	7/27	0·6456	8/23	0·3912	*13/30	0·1679		
2/25	1·4051	5/29	0·9447	6/23	0·6407	7/20	0·3853	10/23	0·1642		
1/12	1·3829	4/23	0·9388	5/19	0·6336	6/17	0·3774	7/16	0·1573		
2/23	1·3597	3/17	0·9289	*4/15	0·6229	5/14	0·3661	11/25	0·1510		
1/11	1·3352	5/28	0·9208	7/26	0·6151	9/25	0·3585	4/9	0·1397		
2/21	1·3092	2/11	0·9085	3/11	0·6046	4/11	0·3488	13/29	0·1300		

Many fractions with higher denominators may be quickly obtained by simple interpolation between adjacent values, *e.g.* from 5/13 and 7/18, because $\frac{5+7}{13+18} = \frac{12}{31}$, the deviate for 12/31 is judged to be $0{\cdot}2822 + \frac{13}{31}(\cdot0112) = 0{\cdot}2868{,}97$. True value 0·2868,94. Each thirtieth is marked by an asterisk.

To convert to probit values subtract from 5, and for fractions λ greater than $\frac{1}{2}$ add the value for $1-\lambda$ to 5.

TABLE XI. PROBITS

Weighting Coefficients and Probit Values to be used in Adjustments of Special Accuracy

(Adapted from Bliss, 1935)

Expected Probit Y.	Minimum Working Probit $Y-P/Z$.	Range $1/Z$	Maximum working Probit $Y+Q/Z$.	Weighting Coefficient Z^2/PQ.	Expected Probit Y.	Minimum Working Probit $Y-P/Z$.	Range $1/Z$	Maximum working Probit $Y+Q/Z$.	Weighting Coefficient Z^2/PQ.
1·1	0·8579	5034	5035	·00082	5·0	3·7467	2·5066	6·2533	·63662
1·2	0·9522	3425	3426	·00118	5·1	3·7401	2·5192	6·2593	·63431
1·3	1·0462	2354	2355	·00167	5·2	3·7186	2·5573	6·2759	·62742
1·4	1·1400	1634	1635	·00235	5·3	3·6798	2·6220	6·3018	·61609
1·5	1·2335	1146	1147	·00327	5·4	3·6203	2·7154	6·3357	·60052
1·6	1·3266	811·5	812·8	·00451	5·5	3·5360	2·8404	6·3764	·58099
1·7	1·4194	580·5	581·9	·00614	5·6	3·4220	3·0010	6·4230	·55788
1·8	1·5118	419·4	420·9	·00828	5·7	3·2724	3·2025	6·4749	·53159
1·9	1·6038	306·1	307·7	·01104	5·8	3·0794	3·4519	6·5313	·50260
2·0	1·6954	225·6	227·3	·01457	5·9	2·8335	3·7582	6·5917	·47144
2·1	1·7866	168·00	169·79	·01903	6·0	2·5230	4·1327	6·6557	·43863
2·2	1·8772	126·34	128·22	·02459	6·1	2·1324	4·5903	6·7227	·40474
2·3	1·9673	95·96	97·93	·03143	6·2	1·6429	5·1497	6·7926	·37031
2·4	2·0568	73·62	75·68	·03977	6·3	1·0295	5·8354	6·8649	·33589
2·5	2·1457	57·05	59·20	·04979	6·4	0·2606	6·6788	6·9394	·30199
2·6	2·2340	44·654	46·888	·06169	6·5	−0·705	7·721	7·0158	·26907
2·7	2·3214	35·302	37·623	·07563	6·6	−1·921	9·015	7·0940	·23753
2·8	2·4081	28·189	30·597	·09179	6·7	−3·459	10·633	7·1739	·20774
2·9	2·4938	22·736	25·230	·11026	6·8	−5·411	12·666	7·2551	·17994
3·0	2·5786	18·522	21·101	·13112	6·9	−7·902	15·240	7·3376	·15436
3·1	2·6624	15·240	17·902	·15436	7·0	−11·101	18·522	7·4214	·13112
3·2	2·7449	12·666	15·411	·17994	7·1	−15·230	22·736	7·5062	·11026
3·3	2·8261	10·633	13·459	·20774	7·2	−20·597	28·189	7·5919	·09179
3·4	2·9060	9·015	11·921	·23753	7·3	−27·623	35·302	7·6786	·07564
3·5	2·9842	7·721	10·705	·26907	7·4	−36·888	44·654	7·7661	·06168
3·6	3·0606	6·6788	9·7394	·30199	7·5	−49·20	57·05	7·8543	·04979
3·7	3·1351	5·8354	8·9705	·33589	7·6	−65·68	73·62	7·9432	·03977
3·8	3·2074	5·1497	8·3571	·37031	7·7	−87·93	95·96	8·0327	·03143
3·9	3·2773	4·5903	7·8676	·40474	7·8	−118·22	126·34	8·1228	·02458
4·0	3·3443	4·1327	7·4770	·43863	7·9	−159·79	168·00	8·2134	·01903
4·1	3·4083	3·7582	7·1665	·47144	8·0	−217·3	225·6	8·3046	·01457
4·2	3·4687	3·4519	6·9206	·50260	8·1	−297·7	306·1	8·3962	·01104
4·3	3·5251	3·2025	6·7276	·53159	8·2	−410·9	419·4	8·4882	·00828
4·4	3·5770	3·0010	6·5780	·55788	8·3	−571·9	580·5	8·5806	·00614
4·5	3·6236	2·8404	6·4640	·58099	8·4	−802·8	811·5	8·6734	·00451
4·6	3·6643	2·7154	6·3797	·60052	8·5	−1137	1146	8·7666	·00327
4·7	3·6982	2·6220	6·3202	·61609	8·6	−1625	1634	8·8600	·00235
4·8	3·7241	2·5573	6·2814	·62741	8·7	−2345	2354	8·9538	·00167
4·9	3·7407	2·5192	6·2599	·63431	8·8	−3416	3425	9·0478	·00118
5·0	3·7467	2·5066	6·2533	·63662	8·9	−5025	5034	9·1421	·00082

In toxicology the probit values used are found by adding 5 to a normal deviate. The weighting coefficient for any test Z^2/PQ is derived from the probit Y given by some provisional formula. The appropriate score for a test in which an observed proportion p die and q survive is y_c, where $y_c = (Y-P/Z)+p/Z = (Y+Q/Z)-q/Z = q(Y-P/Z)+p(Y+Q/Z)$.

TABLE XII. PROBITS

Weighting Coefficients for Use when there is Mortality among the Controls
(D. J. Finney)

Y	Q/Z	Natural Mortality, M												
		·00	·01	·02	·04	·06	·08	·10	·15	·20	·25	·30	·35	·40
1·1	5034	·001	·000											
1·2	3425	·001	·000											
1·3	2354	·002	·000											
1·4	1634	·002	·000											
1·5	1146	·003	·000											
1·6	811·2	·005	·000											
1·7	580·2	·006	·000											
1·8	419·1	·008	·001	·000										
1·9	305·8	·011	·001	·000										
2·0	225·3	·015	·002	·001	·000	·000								
2·1	167·69	·019	·003	·002	·001	·001	·000	·000						
2·2	126·02	·025	·005	·003	·001	·001	·001	·001	·000					
2·3	95·63	·031	·008	·005	·002	·002	·001	·001	·001	·000	·000			
2·4	73·28	·040	·013	·007	·004	·003	·002	·002	·001	·001	·001	·000	·000	
2·5	56·70	·050	·019	·012	·006	·004	·003	·003	·002	·001	·001	·001	·001	·000
2·6	44·288	·062	·028	·018	·010	·007	·005	·004	·003	·002	·001	·001	·001	·001
2·7	34·923	·076	·039	·026	·015	·011	·008	·007	·004	·003	·002	·002	·001	·001
2·8	27·797	·092	·053	·037	·023	·016	·013	·010	·007	·005	·004	·003	·002	·002
2·9	22·330	·110	·070	·051	·033	·024	·019	·015	·010	·007	·006	·004	·004	·003
3·0	18·101	·131	·091	·069	·046	·034	·027	·022	·015	·011	·008	·007	·005	·004
3·1	14·802	·154	·114	·090	·063	·048	·038	·032	·022	·016	·012	·010	·008	·006
3·2	12·211	·180	·140	·115	·083	·065	·053	·044	·030	·023	·018	·014	·011	·009
3·3	10·159	·208	·169	·142	·107	·085	·070	·059	·042	·031	·024	·020	·016	·013
3·4	8·521	·238	·201	·173	·135	·110	·092	·078	·056	·043	·034	·027	·022	·018
3·5	7·205	·269	·234	·206	·166	·138	·117	·101	·074	·057	·045	·036	·030	·025
3·6	6·1394	·302	·268	·241	·199	·169	·145	·127	·095	·074	·059	·048	·039	·033
3·7	5·2705	·336	·304	·277	·235	·202	·177	·156	·119	·094	·076	·062	·051	·043
3·8	4·5571	·370	·340	·315	·272	·238	·211	·188	·146	·117	·095	·078	·065	·055
3·9	3·9676	·405	·377	·352	·310	·275	·247	·222	·176	·142	·117	·097	·081	·068
4·0	3·4770	·439	·412	·389	·347	·313	·283	·258	·208	·170	·141	·119	·100	·084
4·1	3·0665	·471	·447	·424	·384	·350	·320	·294	·241	·200	·168	·142	·120	·102
4·2	2·7206	·503	·480	·458	·420	·386	·356	·330	·274	·231	·195	·166	·142	·121
4·3	2·4276	·532	·510	·490	·454	·421	·391	·364	·307	·261	·224	·192	·165	·142
4·4	2·1780	·558	·538	·519	·484	·453	·424	·397	·339	·292	·252	·218	·188	·163
4·5	1·9640	·581	·563	·545	·512	·481	·453	·427	·370	·321	·279	·243	·212	·184
4·6	1·7797	·601	·583	·567	·536	·507	·480	·454	·397	·348	·305	·268	·234	·205
4·7	1·6202	·616	·600	·585	·556	·528	·502	·477	·421	·372	·329	·290	·256	·224
4·8	1·4814	·627	·613	·598	·571	·545	·520	·496	·442	·394	·350	·311	·275	·243
4·9	1·3599	·634	·621	·607	·582	·557	·534	·511	·458	·411	·368	·328	·292	·259
5·0	1·2533	·637	·624	·612	·588	·565	·542	·521	·471	·424	·382	·343	·307	·273

TABLE XII. PROBITS—*continued*

Weighting Coefficients for Use when there is Mortality among the Controls

Y	Q/Z	Natural Mortality, M.												
		·00	·01	·02	·04	·06	·08	·10	·15	·20	·25	·30	·35	40
5·0	1·2533	·637	·624	·612	·588	·565	·542	·521	·471	·424	·382	·343	·307	·273
5·1	1·1593	·634	·623	·611	·589	·567	·546	·526	·478	·434	·392	·354	·318	·284
5·2	1·0759	·627	·617	·606	·585	·565	·546	·526	·481	·438	·398	·361	·325	·292
5·3	1·0018	·616	·606	·596	·577	·558	·540	·522	·479	·439	·400	·364	·329	·296
5·4	0·9357	·601	·591	·582	·565	·547	·530	·513	·473	·435	·398	·363	·330	·298
5·5	0·8764	·581	·573	·564	·548	·532	·516	·501	·463	·427	·392	·359	·327	·296
5·6	0·8230	·558	·550	·543	·528	·513	·498	·484	·449	·415	·382	·351	·320	·291
5·7	0·7749	·532	·525	·518	·504	·490	·477	·464	·431	·400	·369	·340	·311	·283
5·8	0·7313	·503	·496	·490	·477	·465	·453	·440	·411	·382	·353	·326	·299	·272
5·9	0·6917	·471	·466	·460	·449	·437	·426	·415	·388	·361	·334	·309	·284	·259
6·0	0·6557	·439	·433	·428	·418	·408	·398	·387	·363	·338	·314	·291	·267	·245
6·1	0·6227	·405	·400	·395	·386	·377	·368	·359	·336	·314	·292	·271	·249	·228
6·2	0·5926	·370	·366	·362	·354	·345	·337	·329	·309	·289	·269	·249	·230	·211
6·3	0·5649	·336	·332	·328	·321	·314	·306	·299	·281	·263	·245	·228	·210	·193
6·4	0·5394	·302	·299	·295	·289	·282	·276	·269	·253	·237	·222	·206	·190	·175
6·5	0·5158	·269	·266	·263	·258	·252	·246	·240	·226	·212	·198	·184	·171	·157
6·6	0·4940	·238	·235	·233	·228	·223	·218	·213	·200	·188	·176	·163	·151	·139
6·7	0·4739	·208	·206	·203	·199	·195	·190	·186	·175	·165	·154	·143	·133	·122
6·8	0·4551	·180	·178	·176	·172	·169	·165	·161	·152	·143	·134	·125	·115	·106
6·9	0·4376	·154	·153	·151	·148	·145	·142	·139	·131	·123	·115	·107	·099	·092
7·0	0·4214	·131	·130	·128	·126	·123	·120	·118	·111	·104	·098	·091	·085	·078
7·1	0·4062	·110	·109	·108	·106	·104	·101	·099	·093	·088	·082	·077	·071	·066
7·2	0·3919	·092	·091	·090	·088	·086	·084	·082	·078	·073	·069	·064	·059	·055
7·3	0·3786	·076	·075	·074	·073	·071	·070	·068	·064	·060	·057	·053	·049	·045
7·4	0·3661	·062	·061	·060	·059	·058	·057	·055	·052	·049	·046	·043	·040	·037
7·5	0·3543	·050	·049	·049	·048	·047	·046	·045	·042	·040	·037	·035	·032	·030
7·6	0·3432	·040	·039	·039	·038	·037	·037	·036	·034	·032	·030	·028	·026	·024
7·7	0·3327	·031	·031	·031	·030	·030	·029	·028	·027	·025	·024	·022	·020	·019
7·8	0·3228	·025	·024	·024	·024	·023	·023	·022	·021	·020	·018	·017	·016	·015
7·9	0·3134	·019	·019	·019	·018	·018	·018	·017	·016	·015	·014	·013	·012	·011
8·0	0·3046	·015	·014	·014	·014	·014	·013	·013	·012	·012	·011	·010	·009	·009
8·1	0·2962	·011	·011	·011	·011	·010	·010	·010	·009	·009	·008	·008	·007	·007
8·2	0·2882	·008	·008	·008	·008	·008	·008	·007	·007	·007	·006	·006	·005	·005
8·3	0·2806	·006	·006	·006	·006	·006	·006	·006	·005	·005	·005	·004	·004	·004
8·4	0·2734	·005	·004	·004	·004	·004	·004	·004	·004	·004	·003	·003	·003	·003
8·5	0·2666	·003	·003	·003	·003	·003	·003	·003	·003	·003	·002	·002	·002	·002
8·6	0·2600	·002	·002	·002	·002	·002	·002	·002	·002	·002	·002	·002	·002	·001
8·7	0·2538	·002	·002	·002	·002	·002	·002	·002	·001	·001	·001	·001	·001	·001
8·8	0·2478	·001	·001	·001	·001	·001	·001	·001	·001	·001	·001	·001	·001	·001
8·9	0·2421	·001	·001	·001	·001	·001	·001	·001	·001	·001	·001	·001	·001	·000

TABLE XII. Transformation of Percentages to Degrees

p%	0	1	2	3	4	5	6	7	8	9
0	0	5·7	8·1	10·0	11·5	12·9	14·2	15·3	16·4	17·5
10	18·4	19·4	20·3	21·1	22·0	22·8	23·6	24·4	25·1	25·8
20	26·6	27·3	28·0	28·7	29·3	30·0	30·7	31·3	31·9	32·6
30	33·2	33·8	34·4	35·1	35·7	36·3	36·9	37·5	38·1	38·6
40	39·2	39·8	40·4	41·0	41·6	42·1	42·7	43·3	43·9	44·4
50	45·0	45·6	46·1	46·7	47·3	47·9	48·4	49·0	49·6	50·2
60	50·8	51·4	51·9	52·5	53·1	53·7	54·3	54·9	55·6	56·2
70	56·8	57·4	58·1	58·7	59·3	60·0	60·7	61·3	62·0	62·7
80	63·4	64·2	64·9	65·6	66·4	67·2	68·0	68·9	69·7	70·6
90	71·6	72·5	73·6	74·7	75·8	77·1	78·5	80·0	81·9	84·3

In connection with an observed fraction p, it is often advisable to use the angular value ϕ, such that $p = \sin^2 \phi$. ϕ rises to 90° at 100 per cent. The amount of information supplied by n observations about ϕ, in circular measure, is $4n$, or in degrees, $\frac{4\pi^2 n}{180^2} = \frac{\pi^2 n}{8100}$. Consequently in large samples, the variance of ϕ, as estimated from a sample, is $\frac{8100}{\pi^2 n} = \frac{820{\cdot}7}{n}$ independently of the true value of ϕ.

For small angles use $57{\cdot}3\sqrt{p}$ in preference to interpolation.

TABLE XIII. Transformation of Proper Fractions to Degrees

*1/30	10·5	*1/10	18·4	5/27	25·5	8/29	31·7	*11/30	37·3	9/20	42·1
1/29	10·7	3/29	18·8	3/16	25·7	5/18	31·8	7/19	37·4	5/11	42·4
1/28	10·9	2/19	18·9	4/21	25·9	7/25	31·9	10/27	37·5	11/24	42·6
1/27	11·1	3/28	19·1	5/26	26·0	2/7	32·3	3/8	37·8	6/13	42·8
1/26	11·3	1/9	19·5	*1/5	26·6	7/24	32·7	11/29	38·0	13/28	43·0
1/25	11·5	3/26	19·9	6/29	27·1	5/17	32·8	8/21	38·1	*7/15	43·1
1/24	11·8	2/17	20·1	5/24	27·2	8/27	33·0	5/13	38·3	8/17	43·3
1/23	12·0	3/25	20·3	4/19	27·3	*3/10	33·2	7/18	38·6	9/19	43·5
1/22	12·3	1/8	20·7	3/14	27·6	7/23	33·5	9/23	38·7	10/21	43·6
1/21	12·6	3/23	21·2	5/23	27·8	4/13	33·7	11/28	38·8	11/23	43·8
1/20	12·9	*2/15	21·4	2/9	28·1	9/29	33·9	*2/5	39·2	12/25	43·9
1/19	13·3	3/22	21·7	5/22	28·5	5/16	34·0	11/27	39·7	13/27	43·9
1/18	13·6	4/29	21·8	3/13	28·7	6/19	34·2	9/22	39·8	14/29	44·0
1/17	14·0	1/7	22·2	*7/30	28·9	7/22	34·3	7/17	39·9	*1/2	45·0
1/16	14·5	4/27	22·6	4/17	29·0	8/25	34·4	12/29	40·0		
*1/15	15·0	3/20	22·8	5/21	29·2	9/28	34·5	5/12	40·2		
2/29	15·2	2/13	23·1	6/25	29·3	*1/3	35·3	8/19	40·5		
1/14	15·5	3/19	23·4	7/29	29·4	10/29	36·0	11/26	40·6		
2/27	15·8	4/25	23·6	1/4	30·0	9/26	36·0	3/7	40·9		
1/13	16·1	*1/6	24·1	7/27	30·6	8/23	36·1	*13/30	41·2		
2/25	16·4	5/29	24·5	6/23	30·7	7/20	36·3	10/23	41·3		
1/12	16·8	4/23	24·6	5/19	30·9	6/17	36·4	7/16	41·4		
2/23	17·2	3/17	24·8	*4/15	31·1	5/14	36·7	11/25	41·6		
1/11	17·5	5/28	25·0	7/26	31·3	9/25	36·9	4/9	41·8		
2/21	18·0	2/11	25·2	3/11	31·5	4/11	37·1	13/29	42·0		

For fractions exceeding ½, subtract the fraction from 1, and the angle from 90°. Each thirtieth is marked by an asterisk; for interpolation see Table X.

TABLE XIV. ANGULAR TRANSFORMATION

Angular Values for Adjustments of Special Accuracy

Expected angular value ϕ.	Maximal working angle $\phi + \frac{1}{2}\cot\phi$.	Range cosec 2ϕ.	Expected angular value ϕ.	Maximal working angle $\phi + \frac{1}{2}\cot\phi$.	Range cosec 2ϕ.	Expected angular value ϕ.	Maximal working angle $\phi + \frac{1}{2}\cot\phi$.	Range cosec 2ϕ.
45	73·6	57·3	60	76·5	66·2	75	82·7	114·6
46	73·7	57·3	61	76·9	67·6	76	83·1	122·0
47	73·7	57·4	62	77·2	69·1	77	83·6	130·7
48	73·8	57·6	63	77·6	70·8	78	84·1	140·9
49	73·9	57·9	64	78·0	72·7	79	84·6	152·9
50	74·0	58·3	65	78·4	74·8	80	85·1	167·5
51	74·2	58·6	66	78·8	77·1	81	85·5	185·4
52	74·4	59·0	67	79·2	79·7	82	86·0	207·9
53	74·6	59·6	68	79·6	82·5	83	86·5	236·8
54	74·8	60·2	69	80·0	85·6	84	87·0	275·6
55	75·1	61·0	70	80·4	89·1	85	87·5	330·0
56	75·3	61·8	71	80·9	93·1	86	88·0	411·7
57	75·6	62·7	72	81·3	97·5	87	88·5	548·1
58	75·9	63·7	73	81·8	102·5	88	89·0	821·4
59	76·2	64·9	74	82·2	108·1	89	89·5	1641·7

This table supplies for the angular transformation the same facilities as does Table XI for probits. The weighting coefficient, 1/820·7, is constant for all angular values, and is only needed in testing homogeneity. The adjusted score for *expected* angular value ϕ, is $\phi_e = \phi + \frac{1}{2}\cot\phi - q\ \text{cosec}\ 2\phi$ where q is the observed proportion surviving.

For values less than 45° subtract from 90°; the range is then as tabled; its *lower* extremity is at 90° less the value in the table.

TABLE XIVI. SCORES FOR LINKAGE DATA FROM INTERCROSSES

Recombination fraction. y per cent.	COUPLING			REPULSION		
	Double dominant +	Single recessives −	Double recessive +	Double dominant −	Single recessives +	Double recessive −
1	·6644072	99·4974874	2·0202020	·0099995	·0200020	200·0000000
2	·6620727	49·4949495	2·0408163	·0199960	·0400160	100·0000000
3	·6596620	32·8257191	2·0618557	·0299865	·0600540	66·6666667
4	·6571742	24·4897959	2·0833333	·0399680	·0801282	50·0000000
5	·6546081	19·4871795	2·1052632	·0499376	·1002506	40·0000000
6	·6519628	16·1512027	2·1276596	·0598922	·1204336	33·3333333
7	·6492373	13·7675796	2·1505376	·0698289	·1406894	28·5714286
8	·6464306	11·9791667	2·1739130	·0797448	·1610306	25·0000000
9	·6435416	10·5875509	2·1978022	·0896370	·1814699	22·2222222
10	·6405694	9·4736842	2·2222222	·0995025	·2020202	20·0000000
11	·6375130	8·5618086	2·2471910	·1093385	·2226946	18·1818182
12	·6343714	7·8014184	2·2727273	·1191422	·2435065	16·6666667
13	·6311437	7·1575483	2·2988506	·1289107	·2644695	15·3846154
14	·6278289	6·6052227	2·3255814	·1386413	·2855977	14·2857143
15	·6244261	6·1261261	2·3529412	·1483313	·3069054	13·3333333
16	·6209344	5·7065217	2·3809524	·1579779	·3284072	12·5000000
17	·6173528	5·3359049	2·4096386	·1675785	·3501184	11·7647059
18	·6136806	5·0061050	2·4390244	·1771305	·3720546	11·1111111
19	·6099168	4·7106717	2·4691358	·1866313	·3942318	10·5263158
20	·6060606	4·4444444	2·5000000	·1960784	·4166667	10·0000000
21	·6021112	4·2032455	2·5316456	·2054694	·4393765	9·5238095
22	·5980678	3·9836568	2·5641026	·2148018	·4623792	9·0909091
23	·5939296	3·7828543	2·5974026	·2240733	·4856932	8·6956522
24	·5896958	3·5984848	2·6315789	·2332815	·5093379	8·3333333
25	·5853659	3·4285714	2·6666667	·2424242	·5333333	8·0000000

TABLE XIVI. SCORES FOR LINKAGE DATA FROM INTERCROSSES—*continued*

Recombination fraction. y per cent.	COUPLING			REPULSION		
	Double dominant +	Single recessives −	Double recessive +	Double dominant −	Single recessives +	Double recessive −
26	·5809389	3·2714412	2·7027027	·2514993	·5577006	7·6923077
27	·5764144	3·1256690	2·7397260	·2605046	·5824614	7·4074074
28	·5717916	2·9900332	2·7777778	·2694380	·6076389	7·1428571
29	·5670700	2·8634805	2·8169014	·2782976	·6332569	6·8965517
30	·5622490	2·7450980	2·8571429	·2870813	·6593407	6·6666667
31	·5573281	2·6340905	2·8985507	·2957874	·6859166	6·4516129
32	·5523067	2·5297619	2·9411765	·3044140	·7130125	6·2500000
33	·5471845	2·4315006	2·9850746	·3129594	·7406576	6·0606061
34	·5419609	2·3387668	3·0303030	·3214218	·7688829	5·8823529
35	·5366357	2·2510823	3·0769231	·3297998	·7977208	5·7142857
36	·5312085	2·1680217	3·1250000	·3380917	·8272059	5·5555556
37	·5256790	2·0892058	3·1746032	·3462960	·8573746	5·4054054
38	·5200470	2·0142950	3·2258065	·3544115	·8882655	5·2631579
39	·5143122	1·9429846	3·2786885	·3624367	·9199198	5·1282051
40	·5084746	1·8750000	3·3333333	·3703704	·9523810	5·0000000
41	·5025340	1·8100936	3·3898305	·3782113	·9856954	4·8780488
42	·4964903	1·7480410	3·4482759	·3859585	1·0199126	4·7619048
43	·4903437	1·6886387	3·5087719	·3936107	1·0550853	4·6511628
44	·4840941	1·6317016	3·5714286	·4011670	1·0912698	4·5454545
45	·4777416	1·5770609	3·6363636	·4086266	1·1285266	4·4444444
46	·4712864	1·5245624	3·7037037	·4159884	1·1669203	4·3478261
47	·4647288	1·4740648	3·7735849	·4232518	1·2065203	4·2553191
48	·4580691	1·4254386	3·8461538	·4304161	1·2474012	4·1666667
49	·4513075	1·3785647	3·9215686	·4374805	1·2896434	4·0816327
50	·4444444	1·3333333	4·0000000	·4444444	1·3333333	4·0000000

Table XV. Latin Squares

The 4×4 Latin Squares

First Transformation Set: 3 Self-Conjugate Standard Squares

A B C D B A D C C D B A D C A B	A B C D B C D A C D A B D A B C	A B C D B D A C C A D B D C B A
1	2	3

Second Transformation Set: 1 Self-Conjugate Standard Square

A B C D B A D C C D A B D C B A
4

The 5×5 Latin Squares

First Transformation Set: 25 Standard Squares and their Conjugates

A B C D E B A E C D C D A E B D E B A C E C D B A	A B C D E B A D E C C E B A D D C E B A E D A C B	A B C D E B A E C D C E D A B D C B E A E D A B C	A B C D E B A D E C C D E A B D E B C A E C A B D	A B C D E B C D E A C E B A D D A E B C E D A C B
1, 2	3, 4	5, 6	7, 8	9, 10
A B C D E B C D E A C E A B D D A E C B E D B A C	A B C D E B C E A D C D B E A D E A C B E A D B C	A B C D E B C E A D C A D E B D E B C A E D A B C	A B C D E B C D E A C A E B D D E A C B E D B A C	A B C D E B C A E D C E D A B D A E B C E D B C A
11, 12	13, 14	15, 16	17, 18	19, 20
A B C D E B C A E D C D E B A D E B A C E A D C B	A B C D E B D E C A C A B E D D E A B C E C D A B	A B C D E B D A E C C E D B A D C E A B E A B C D	A B C D E B D E C A C A D E B D E A B C E C B A D	A B C D E B D A E C C E D B A D A E C B E C B A D
21, 22	23, 24	25, 26	27, 28	29, 30
A B C D E B D E A C C E D B A D C A E B E A B C D	A B C D E B D A E C C E B A D D A E C B E C D B A	A B C D E B D E A C C E A B D D C B E A E A D C B	A B C D E B E D A C C A B E D D C E B A E D A C B	A B C D E B E D A C C A E B D D C A E B E D B C A
31, 32	33, 34	35, 36	37, 38	39, 40
A B C D E B E A C D C D E B A D A B E C E C D A B	A B C D E B E D C A C A E B D D C A E B E D B A C	A B C D E B E D C A C D E A B D A B E C E C A B D	A B C D E B E A C D C D B E A D C E A B E A D B C	A B C D E B E D A C C D A E B D C E B A E A B C D
41, 42	43, 44	45, 46	47, 48	49, 50

Second Transformation Set: 6 Self-Conjugate Standard Squares

A B C D E B C E A D C E D B A D A B E C E D A C B	A B C D E B C D E A C D E A B D E A B C E A B C D	A B C D E B D E C A C E B A D D C A E B E A D B C	A B C D E B D A E C C A E B D D E B C A E C D A B	A B C D E B E D A C C D B E A D A E C B E C A B D	A B C D E B E A C D C A D E B D C E B A E D B A C
51	52	53	54	55	56

When constructing squares for use in experimental arrangements use the randomization procedure described in the Introduction.

TABLE XV. LATIN SQUARES—*continued*

The 6×6 Latin Squares

Five conjugate pairs of transformation sets, and twelve sets containing conjugates. The numbers above the squares indicate the number in each set, *s* standing for self-conjugate and *c* for conjugate. Each pair of conjugate sets can be turned into the following transformation set by interchanging rows or columns with letters.

I

1080, 1080

A	B	C	D	E	F
B	C	F	A	D	E
C	F	B	E	A	D
D	E	A	B	F	C
E	A	D	F	C	B
F	D	E	C	B	A

0001-1080
1081-2160

II

180*s*+450+450*c*

A	B	C	D	E	F
B	C	F	E	A	D
C	F	B	A	D	E
D	E	A	B	F	C
E	A	D	F	C	B
F	D	E	C	B	A

2161-3240

III

540+540*c*

A	B	C	D	E	F
B	A	F	E	C	D
C	F	B	A	D	E
D	C	E	B	F	A
E	D	A	F	B	C
F	E	D	C	A	B

3241-4320

IV

540+540*c*

A	B	C	D	E	F
B	A	E	F	C	D
C	F	B	A	D	E
D	E	A	B	F	C
E	D	F	C	B	A
F	C	D	E	A	B

4321-5400

V

540, 540

A	B	C	D	E	F
B	A	E	C	F	D
C	F	B	A	D	E
D	E	F	B	C	A
E	D	A	F	B	C
F	C	D	E	A	B

5401-5940
5941-6480

VI

90*s*+225+225*c*

A	B	C	D	E	F
B	A	F	E	C	D
C	F	B	A	D	E
D	E	A	B	F	C
E	C	D	F	B	A
F	D	E	C	A	B

6481-7020

VII

270+270*c*

A	B	C	D	E	F
B	C	D	E	F	A
C	E	A	F	B	D
D	F	B	A	C	E
E	D	F	B	A	C
F	A	E	C	D	B

7021-7560

VIII

360, 360

A	B	C	D	E	F
B	A	E	F	C	D
C	F	A	E	D	B
D	C	B	A	F	E
E	D	F	C	B	A
F	E	D	B	A	C

7561-7920
7921-8280

IX

180+180*c*

A	B	C	D	E	F
B	A	E	F	C	D
C	F	A	B	D	E
D	E	B	A	F	C
E	D	F	C	B	A
F	C	D	E	A	B

8281-8640

X

60*s*+60+60*c*

A	B	C	D	E	F
B	C	F	A	D	E
C	F	B	E	A	D
D	A	E	B	F	C
E	D	A	F	C	B
F	E	D	C	B	A

8641-8820

XI

120, 120

A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	F	E	B	A	C
E	D	F	C	B	A
F	E	D	A	C	B

8821-8940
8941-9060

XII

60*s*+30+30*c*

A	B	C	D	E	F
B	C	A	E	F	D
C	A	B	F	D	E
D	E	F	B	A	C
E	F	D	A	C	B
F	D	E	C	B	A

9061-9180

XIII

60*s*

A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	F	E	B	A	C
E	D	F	A	C	B
F	E	D	C	B	A

9181-9240

XIV

20+20*c*

A	B	C	D	E	F
B	C	A	E	F	D
C	A	B	F	D	E
D	F	E	B	A	C
E	D	F	C	B	A
F	E	D	A	C	B

9241-9280

XV

36, 36

A	B	C	D	E	F
B	A	F	E	D	C
C	D	A	B	F	E
D	F	E	A	C	B
E	C	B	F	A	D
F	E	D	C	B	A

9281-9316
9317-9352

XVI

6*s*+15+15*c*

A	B	C	D	E	F
B	A	E	C	F	D
C	E	A	F	D	B
D	C	F	A	B	E
E	F	D	B	A	C
F	D	B	E	C	A

9353-9388

XVII

10+10*c*

A	B	C	D	E	F
B	C	A	F	D	E
C	A	B	E	F	D
D	E	F	A	B	C
E	F	D	C	A	B
F	D	E	B	C	A

9389-9408

Examples of 7×7 Latin Squares

A	B	C	D	E	F	G
B	E	A	G	F	D	C
C	F	G	B	D	A	E
D	G	E	F	C	B	A
E	D	B	C	A	G	F
F	C	D	A	G	E	B
G	A	F	E	B	C	D

(*a*)

A	B	C	D	E	F	G
B	E	A	G	F	D	C
C	F	G	B	D	A	E
D	G	E	F	B	C	A
E	D	B	C	A	G	F
F	C	D	A	G	E	B
G	A	F	E	C	B	D

(*b*)

A_1	B_2	C_3	D_4	E_5	F_6	G_7
B_3	F_7	E_6	G_1	C_4	A_2	D_5
C_7	D_1	A_5	E_2	B_6	G_4	F_3
D_6	C_5	G_2	A_3	F_1	E_7	B_4
E_4	G_6	B_1	F_5	A_7	D_3	C_2
F_2	A_4	D_7	C_6	G_3	B_5	E_1
G_5	E_3	F_4	B_7	D_2	C_1	A_6

(*c*)

A_1	B_2	C_3	D_4	E_5	F_6	G_7
B_3	C_7	D_6	E_1	F_4	G_2	A_5
C_6	D_5	E_2	F_3	G_1	A_7	B_4
D_2	E_4	F_7	G_6	A_3	B_5	C_1
E_7	F_1	G_5	A_2	B_6	C_4	D_3
F_5	G_3	A_4	B_7	C_2	D_1	E_6
G_4	A_6	B_1	C_5	D_7	E_3	F_2

(*d*)

When constructing squares for use in experimental arrangements use the randomization procedure described in the Introduction.

TABLE XV. LATIN SQUARES—*continued*

Examples of Squares from 8×8 to 12×12

A	B	C	D	E	F	G	H
B	C	A	E	F	D	H	G
C	A	D	G	H	E	F	B
D	F	G	C	A	H	B	E
E	H	B	F	G	C	A	D
F	D	H	A	B	G	E	C
G	E	F	H	C	B	D	A
H	G	E	B	D	A	C	F

8×8

A	B	C	D	E	F	G	H	I
B	C	E	G	D	I	F	A	H
C	D	F	A	H	G	I	E	B
D	H	A	B	F	E	C	I	G
E	G	B	I	C	H	D	F	A
F	I	H	E	B	D	A	G	C
G	F	I	C	A	B	H	D	E
H	E	G	F	I	A	B	C	D
I	A	D	H	G	C	E	B	F

9×9

A	B	C	D	E	F	G	H	I	J
B	G	A	E	H	C	F	I	J	D
C	H	J	G	F	B	E	A	D	I
D	A	G	I	J	E	C	B	F	H
E	F	H	J	I	G	A	D	B	C
F	E	B	C	D	I	J	G	H	A
G	I	F	B	A	D	H	J	C	E
H	C	I	F	G	J	D	E	A	B
I	J	D	A	C	H	B	F	E	G
J	D	E	H	B	A	I	C	G	F

10×10

A	B	C	D	E	F	G	H	I	J	K
B	A	J	I	D	C	F	K	H	G	E
C	K	H	A	B	I	J	F	D	E	G
D	C	G	J	I	K	E	B	F	A	H
E	J	B	G	K	H	D	C	A	I	F
F	E	I	C	G	A	K	J	B	H	D
G	F	D	B	H	J	A	I	E	K	C
H	I	K	F	A	D	B	E	G	C	J
I	D	E	H	J	B	C	G	K	F	A
J	G	A	K	F	E	H	D	C	B	I
K	H	F	E	C	G	I	A	J	D	B

11×11

A	B	C	D	E	F	G	H	I	J	K	L
B	L	G	C	D	J	K	E	H	A	F	I
C	K	A	B	F	L	I	D	G	H	J	E
D	F	I	A	L	E	C	G	J	B	H	K
E	D	F	G	J	K	A	L	C	I	B	H
F	H	K	E	G	C	D	B	A	L	I	J
G	I	D	F	K	H	J	A	L	C	E	B
H	E	L	J	C	A	B	I	K	D	G	F
I	J	B	L	H	G	F	K	D	E	A	C
J	C	E	K	A	I	H	F	B	G	L	D
K	G	J	H	I	B	L	C	E	F	D	A
L	A	H	I	B	D	E	J	F	K	C	G

12×12

TABLE XVI. COMPLETE SETS OF ORTHOGONAL LATIN SQUARES

3×3

I

1	2	3
2	3	1
3	1	2

II

1	2	3
3	1	2
2	3	1

4×4

I

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

AC, BD

II

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

ACD, BC

III

1	2	3	4
4	3	2	1
2	1	4	3
3	4	1	2

AD, BCD

5×5

I

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

II

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

III

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

IV

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

7×7

I

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

II

1	2	3	4	5	6	7
3	4	5	6	7	1	2
5	6	7	1	2	3	4
7	1	2	3	4	5	6
2	3	4	5	6	7	1
4	5	6	7	1	2	3
6	7	1	2	3	4	5

III

1	2	3	4	5	6	7
4	5	6	7	1	2	3
7	1	2	3	4	5	6
3	4	5	6	7	1	2
6	7	1	2	3	4	5
2	3	4	5	6	7	1
5	6	7	1	2	3	4

IV

1	2	3	4	5	6	7
5	6	7	1	2	3	4
2	3	4	5	6	7	1
6	7	1	2	3	4	5
3	4	5	6	7	1	2
7	1	2	3	4	5	6
4	5	6	7	1	2	3

V

1	2	3	4	5	6	7
6	7	1	2	3	4	5
4	5	6	7	1	2	3
2	3	4	5	6	7	1
7	1	2	3	4	5	6
5	6	7	1	2	3	4
3	4	5	6	7	1	2

VI

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

8×8

I

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

AD, BE, CF

II

1	2	3	4	5	6	7	8
5	6	7	8	1	2	3	4
2	1	4	3	6	5	8	7
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
8	7	6	5	4	3	2	1
4	3	2	1	8	7	6	5

AEF, BD, CE

III

1	2	3	4	5	6	7	8
7	8	5	6	3	4	1	2
5	6	7	8	1	2	3	4
3	4	1	2	7	8	5	6
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
4	3	2	1	8	7	6	5
6	5	8	7	2	1	4	3

ADE, BEF, CD

IV

1	2	3	4	5	6	7	8
8	7	6	5	4	3	2	1
7	8	5	6	3	4	1	2
2	1	4	3	6	5	8	7
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
3	4	1	2	7	8	5	6

ADEF, BDE, CEF

V

1	2	3	4	5	6	7	8
4	3	2	1	8	7	6	5
8	7	6	5	4	3	2	1
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
2	1	4	3	6	5	8	7

ADF, BDEF, CDE

VI

1	2	3	4	5	6	7	8
6	5	8	7	2	1	4	3
4	3	2	1	8	7	6	5
7	8	5	6	3	4	1	2
3	4	1	2	7	8	5	6
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
5	6	7	8	1	2	3	4

AF, BDF, CDEF

VII

1	2	3	4	5	6	7	8
3	4	1	2	7	8	5	6
6	5	8	7	2	1	4	3
8	7	6	5	4	3	2	1
2	1	4	3	6	5	8	7
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
7	8	5	6	3	4	1	2

AE, BF, CDF

9×9

I

1	2	3	4	5	6	7	8	9
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5

AC(J), BD(J), AC(J)×BD(J).

II

1	2	3	4	5	6	7	8	9
7	8	9	1	2	3	4	5	6
4	5	6	7	8	9	1	2	3
2	3	1	5	6	4	8	9	7
8	9	7	2	3	1	5	6	4
5	6	4	8	9	7	2	3	1
3	1	2	6	4	5	9	7	8
9	7	8	3	1	2	6	4	5
6	4	5	9	7	8	3	1	2

BC(J), AD(I), BC(J)×AD(I).

III

1	2	3	4	5	6	7	8	9
9	7	8	3	1	2	6	4	5
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2
2	3	1	5	6	4	8	9	7
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
4	5	6	7	8	9	1	2	3
3	1	2	6	4	5	9	7	8

ABC(Y), ABD(W), BCD(Y), ACD(Z).

IV

1	2	3	4	5	6	7	8	9
8	9	7	2	3	1	5	6	4
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5
4	5	6	7	8	9	1	2	3
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
3	1	2	6	4	5	9	7	8
7	8	9	1	2	3	4	5	6

ABC(X), ABD(Y), ACD(X), BCD(Z).

V.

1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
2	3	1	5	6	4	8	9	7
7	8	9	1	2	3	4	5	6
9	7	8	3	1	2	6	4	5
8	9	7	2	3	1	5	6	4
4	5	6	7	8	9	1	2	3
6	4	5	9	7	8	3	1	2
5	6	4	8	9	7	2	3	1

AC(I), BD(I), AC(I)×BD(I).

VI

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4

BC(I), AD(J), BC(I)×AD(J).

VII

1	2	3	4	5	6	7	8	9
5	6	4	8	9	7	2	3	1
9	7	8	3	1	2	6	4	5
8	9	7	2	3	1	5	6	4
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	1	2	3
6	4	5	9	7	8	3	1	2
7	8	9	1	2	3	4	5	6
2	3	1	5	6	4	8	9	7

ABC(Z), ABD(X), BCD(X), ACD(W).

VIII

1	2	3	4	5	6	7	8	9
6	4	5	9	7	8	3	1	2
8	9	7	2	3	1	5	6	4
5	6	4	8	9	7	2	3	1
7	8	9	1	2	3	4	5	6
3	1	2	6	4	5	9	7	8
9	7	8	3	1	2	6	4	5
2	3	1	5	6	4	8	9	7
4	5	6	7	8	9	1	2	3

ABC(W), ABD(Z), ACD(Y), BCD(W).

TABLE XVII. Combinatorial Solutions

1 $r\ 5\ \ k\ 3$ — $v\ 6\ \ b\ 10$ — $\lambda\ 2$

Use 2, deleting any set of five varieties occurring in the same block

2 $r\ k\ 5\ \ \lambda\ 2$ — $v\ b\ 11$

Cyclic solution

$a\ e\ f\ g\ i$

3 $r\ 6\ \ k\ 4$ — $v\ 10\ \ b\ 15$ — $\lambda\ 2$

Use 5, deleting any set of six varieties occurring in the same block

4 $r\ 6\ \ k\ 3$ — $v\ 13\ \ b\ 26$ — $\lambda\ 1$

Cyclic solution in two families

$a\ c\ i$

$b\ f\ e$

5 $r\ k\ 6$ — $v\ b\ 16$ — $\lambda\ 2$

Dicyclic solution

$a_1\ a_2\ a_3\ b_1\ c_4\ d_1$

6† $r\ 7$ — $v\ 8$ — $k\ 4$ — $b\ 14$ — $\lambda\ 3$

$a\ b\ c\ d$	$a\ b\ g\ h$	$a\ c\ f\ h$	$a\ d\ f\ g$
$e\ f\ g\ h$	$c\ d\ e\ f$	$b\ d\ e\ g$	$b\ c\ e\ h$
$a\ b\ e\ f$	$a\ c\ e\ g$	$a\ d\ e\ h$	
$c\ d\ g\ h$	$b\ d\ f\ h$	$b\ c\ f\ g$	

7† $r\ 7$ — $v\ 15$ — $k\ 3$ — $b\ 35$ — $\lambda\ 1$

$a\ b\ c$	$a\ d\ e$	$a\ f\ g$	$a\ h\ i$	$a\ j\ k$	$a\ l\ m$
$d\ h\ l$	$b\ h\ j$	$b\ i\ k$	$b\ m\ o$	$b\ l\ n$	$b\ e\ g$
$e\ j\ o$	$c\ m\ n$	$c\ l\ o$	$c\ d\ g$	$c\ e\ f$	$c\ i\ j$
$f\ k\ m$	$f\ i\ o$	$d\ j\ n$	$e\ k\ n$	$d\ i\ m$	$d\ k\ o$
$g\ i\ n$	$g\ k\ l$	$e\ h\ m$	$f\ j\ l$	$g\ h\ o$	$f\ h\ n$
$a\ n\ o$	$b\ d\ f$	$c\ h\ k$	$e\ i\ l$	$g\ j\ m$	

9 $r\ k\ 7\ \ \lambda\ 3$ — $v\ b\ 15$

Cyclic solution

$a\ b\ c\ e\ f\ i\ k$

11 $r\ 8\ \ k\ 4$ — $v\ 9\ \ b\ 18$ — $\lambda\ 3$

Use 19, taking the first four varieties of each block except the first

13 $r\ 8\ \ k\ 4$ — $v\ 25\ \ b\ 50$ — $\lambda\ 1$

Dicyclic solution in two families

$a_1\ a_2\ b_1\ e_5$

$a_1\ a_3\ c_1\ d_4$

15 $r\ 9$ — $v\ 10$ — $k\ 3$ — $b\ 30$ — $\lambda\ 2$

$a\ b\ c$	$a\ g\ i$	$b\ e\ i$	$c\ e\ f$	$d\ f\ i$
$a\ b\ d$	$a\ h\ j$	$b\ f\ g$	$c\ g\ j$	$d\ g\ h$
$a\ c\ e$	$a\ i\ j$	$b\ g\ i$	$c\ h\ i$	$e\ f\ j$
$a\ d\ f$	$b\ c\ f$	$b\ h\ j$	$c\ i\ j$	$e\ g\ h$
$a\ e\ g$	$b\ d\ j$	$c\ d\ g$	$d\ e\ i$	$f\ g\ j$
$a\ f\ h$	$b\ e\ h$	$c\ d\ h$	$d\ e\ j$	$f\ h\ i$

16 $r\ 9\ \ k\ 5$ — $v\ 10\ \ b\ 18$ — $\lambda\ 4$

Use 19, deleting any set of nine varieties occurring in the same block

17 $r\ 9\ \ k\ 6$ — $v\ 16\ \ b\ 24$ — $\lambda\ 3$

Use 20, deleting any set of nine varieties occurring in the same block

18 $r\ 9\ \ k\ 3$ — $v\ 19\ \ b\ 57$ — $\lambda\ 1$

Cyclic solution in three families

$a\ g\ k,\ \ b\ c\ n,\ \ d\ f\ i$

19 $r\ k\ 9\ \ \lambda\ 4$ — $v\ b\ 19$

Cyclic solution

$a\ c\ e\ f\ g\ h\ k\ n\ o$

20 $r\ k\ 9$ — $v\ b\ 25$ — $\lambda\ 3$

$a\ b\ c\ d\ e\ f\ g\ h\ i$	$b\ f\ g\ j\ l\ r\ t\ w\ y$
$a\ b\ c\ n\ o\ r\ s\ v\ w$	$b\ f\ g\ k\ m\ s\ u\ v\ x$
$a\ b\ c\ p\ q\ t\ u\ x\ y$	$c\ d\ h\ j\ m\ r\ u\ w\ x$
$a\ d\ g\ j\ k\ n\ o\ t\ u$	$c\ d\ h\ k\ l\ s\ t\ v\ y$
$a\ d\ g\ l\ m\ p\ q\ r\ s$	$c\ e\ g\ j\ m\ o\ p\ v\ y$
$a\ e\ i\ j\ k\ r\ s\ x\ y$	$c\ e\ g\ k\ l\ n\ q\ w\ x$
$a\ e\ i\ l\ m\ t\ u\ v\ w$	$c\ f\ i\ j\ m\ n\ q\ s\ t$
$a\ f\ h\ j\ k\ p\ q\ v\ w$	$c\ f\ i\ k\ l\ o\ p\ r\ u$
$a\ f\ h\ l\ m\ n\ o\ x\ y$	$d\ e\ f\ n\ p\ r\ t\ v\ x$
$b\ d\ i\ j\ l\ o\ q\ v\ x$	$d\ e\ f\ o\ q\ s\ u\ w\ y$
$b\ d\ i\ k\ m\ n\ p\ w\ y$	$g\ h\ i\ n\ q\ r\ u\ v\ y$
$b\ e\ h\ j\ l\ n\ p\ s\ u$	$g\ h\ i\ o\ p\ s\ t\ w\ x$
$b\ e\ h\ k\ m\ o\ q\ r\ t$	

21† $r\ 9$ — $v\ 28$ — $k\ 4$ — $b\ 63$ — $\lambda\ 1$

Invariant variety I, and 27 others $a_1^1 \ldots a_3^3$, $b_1^1 \ldots b_3^3$, $c_1^1 \ldots c_3^3$. Blocks $a_2^1\ a_3^1\ b_1^2\ b_1^3$, $a_2^3\ b_2^2\ a_3^2\ b_3^3$, I $a_1^1\ b_1^1\ c_1^1$. Cyclic permutation $(a\ b\ c)$ generates 7 blocks in a replication; (123) in suffixes and indices generates the 9 replications.

22 $r\ 9\ \ k\ 7$ — $v\ 28\ \ b\ 36$ — $\lambda\ 2$

Use 23, deleting any set of nine varieties occurring in the same block

23 $r\ k\ 9\ \ \lambda\ 2$ — $v\ b\ 37$

Cyclic solution

$a\ b\ d\ h\ r\ y\ z\ D\ J$

25† $r\ 10$ — $v\ 21$ — $k\ 3$ — $b\ 70$ — $\lambda\ 1$

$a\ b\ c$	$d\ e\ f$	$g\ h\ i$	$j\ k\ l$	$m\ n\ o$	$p\ q\ r$	$s\ t\ u$		
$a\ d\ o$	$b\ e\ k$	$c\ i\ p$	$f\ q\ t$	$g\ l\ s$	$h\ m\ r$	$j\ n\ u$		
$a\ e\ q$	$b\ d\ n$	$c\ g\ k$	$f\ j\ s$	$h\ p\ t$	$i\ o\ r$	$l\ m\ u$		
$a\ f\ i$	$b\ g\ p$	$c\ h\ u$	$d\ q\ s$	$e\ j\ m$	$k\ o\ t$	$l\ n\ r$		
$a\ g\ u$	$b\ m\ q$	$c\ j\ r$	$d\ h\ k$	$e\ p\ s$	$f\ l\ o$	$i\ n\ t$		
$a\ h\ j$	$b\ r\ s$	$c\ o\ q$	$d\ l\ p$	$e\ i\ u$	$f\ k\ n$	$g\ m\ t$		
$a\ k\ r$	$b\ j\ t$	$c\ e\ l$	$d\ i\ m$	$f\ p\ u$	$g\ n\ q$	$h\ o\ s$		
$a\ l\ t$	$b\ f\ h$	$c\ n\ s$	$d\ r\ u$	$e\ g\ o$	$i\ j\ q$	$k\ m\ p$		
$a\ m\ s$	$b\ i\ l$	$c\ d\ t$	$e\ h\ n$	$f\ g\ r$	$j\ o\ p$	$k\ q\ u$		
$a\ n\ p$	$b\ o\ u$	$c\ f\ m$	$d\ g\ j$	$e\ r\ t$	$h\ l\ q$	$i\ k\ s$		

26 $r\ 10\ \ k\ 7\ \ \lambda\ 3$ — $v\ 21\ \ b\ 30$

Use 27, deleting any set of ten varieties occurring in the same block

27 $r\ k\ 10$ — $v\ b\ 31$ — $\lambda\ 3$

$a_1\ a_6\ b_2\ b_5\ c_3\ c_4\ d_3\ d_5\ d_6\ e$	Cyclic permutation (1234567)
$a_2\ a_5\ b_3\ b_4\ c_1\ c_6\ d_3\ d_5\ d_6\ f$	
$a_3\ a_4\ b_1\ b_6\ c_2\ c_5\ d_3\ d_5\ d_6\ g$	
$a_1\ a_2\ a_4\ b_1\ b_2\ b_4\ c_1\ c_2\ c_4\ d_7$	
$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ e\ f\ g$	Permutation $(a\ b\ c)$

29 $r\ 10\ \ k\ 5$ — $v\ 41\ \ b\ 82$ — $\lambda\ 1$

Cyclic solution in two families

$a\ j\ p\ r\ K$

$a\ s\ E\ F\ I$

† Divisible into groups, each containing a complete replication, No. 25 by rows, Nos. 6, 7 and 21 as indicated.

TABLE XVIII. Index by Number of Replications

Replications (r).	Varieties (v).	Blocks (b).	Units per block (k).	Reference.
3	4	6	2	*u*
	4	4	3	*u*
	7	7	3	*o.s.*
4	5	10	2	*u*
	5	5	4	*u*
	7	7	4	*o.s.*
	9	12	3	*o.s.*
	13	13	4	*o.s.*
5	6	15	2	*u*
	6	10	3	1
	6	6	5	*u*
	11	11	5	2
	16	20	4	*o.s.*
	21	21	5	*o.s.*
6	5	10	3	*u*
	7	21	2	*u*
	7	7	6	*u*
	10	15	4	3
	11	11	6	2
	13	26	3	4
	16	16	6	5
	25	30	5	*o.s.*
	31	31	6	*o.s.*
7	8	28	2	*u*
	8	14	4	6
	8	8	7	*u*
	15	35	3	7
	15	21	5	*
	15	15	7	9
	22	22	7	*
8	9	36	2	*u*
	9	18	4	11
	9	12	6	*o.s.*
	9	9	8	*u*
	15	15	8	9
	21	28	6	—
	25	50	4	13
	29	29	8	*
	49	56	7	*o.s.*
	57	57	8	*o.s.*
9	10	45	2	*u*
	10	30	3	15
	10	18	5	16
	10	15	6	3
	10	10	9	*u*
	13	13	9	*o.s.*
	16	24	6	17
9	19	57	3	18
	19	19	9	19
	25	25	9	20
	28	63	4	21
	28	36	7	22
	37	37	9	23
	46	69	6	—
	64	72	8	*o.s.*
	73	73	9	*o.s.*
10	6	15	4	*u*
	9	18	5	11
	11	55	2	*u*
	11	11	10	*u*
	16	16	10	5
	19	19	10	19
	21	70	3	25
	21	30	7	26
	31	31	10	27
	36	45	8	—
	41	82	5	29
	46	46	10	—
	51	85	6	—
	81	90	9	*o.s.*
	91	91	10	*o.s.*

TABLE XIX. Index by Number of Units in a Block

Units per block (k).	Blocks (b).	Varieties (v).	Replications (r).	Reference.
3	4	4	3	*u*
	7	7	3	*o.s.*
	10	5	6	*u*
	10	6	5	1
	12	9	4	*o.s.*
	26	13	6	4
	30	10	9	15
	35	15	7	7
	57	19	9	18
	70	21	10	25
4	5	5	4	*u*
	7	7	4	*o.s.*
	13	13	4	*o.s.*
	14	8	7	6
	15	6	10	*u*
	15	10	6	3
	18	9	8	11
	20	16	5	*o.s.*
	50	25	8	13
	63	28	9	21
5	6	6	5	*u*
	11	11	5	2
5	18	9	10	11
	18	10	9	16
	21	15	7	*
	21	21	5	*o.s.*
	30	25	6	*o.s.*
	82	41	10	29
6	7	7	6	*u*
	11	11	6	2
	12	9	8	*o.s.*
	15	10	9	3
	16	16	6	5
	24	16	9	17
	28	21	8	—
	31	31	6	*o.s.*
	69	46	9	—
	85	51	10	—
7	8	8	7	*u*
	15	15	7	9
	22	22	7	*
	30	21	10	26
	36	28	9	22
	56	49	8	*o.s.*
8	9	9	8	*u*
	15	15	8	9
	29	29	8	*
	45	36	10	—
	57	57	8	*o.s.*
	72	64	9	*o.s.*
9	10	10	9	*u*
	13	13	9	*o.s.*
	19	19	9	19
	25	25	9	20
	37	37	9	23
	73	73	9	*o.s.*
	90	81	10	*o.s.*
10	11	11	10	*u*
	16	16	10	5
	19	19	10	19
	31	31	10	27
	46	46	10	—
	91	91	10	*o.s.*

Table XX. Scores for Ordinal (or Ranked) Data

The mean deviations of the 1st, 2nd, 3rd . . . largest members of samples of different sizes; zero and negative values omitted.

Ordinal number.	Size of Sample									
	—	2	3	4	5	6	7	8	9	10
1		·56	·85	1·03	1·16	1·27	1·35	1·42	1·49	1·54
2				·30	·50	·64	·76	·85	·93	1·00
3						·20	·35	·47	·57	·66
4								·15	·27	·38
5										·12
	11	12	13	14	15	16	17	18	19	20
1	1·59	1·63	1·67	1·70	1·74	1·76	1·79	1·82	1·84	1·87
2	1·06	1·12	1·16	1·21	1·25	1·28	1·32	1·35	1·38	1·41
3	·73	·79	·85	·90	·95	·99	1·03	1·07	1·10	1·13
4	·46	·54	·60	·66	·71	·76	·81	·85	·89	·92
5	·22	·31	·39	·46	·52	·57	·62	·67	·71	·75
6		·10	·19	·27	·34	·39	·45	·50	·55	·59
7				·09	·17	·23	·30	·35	·40	·45
8						·08	·15	·21	·26	·31
9								·07	·13	·19
10										·06
	21	22	23	24	25	26	27	28	29	30
1	1·89	1·91	1·93	1·95	1·97	1·98	2·00	2·01	2·03	2·04
2	1·43	1·46	1·48	1·50	1·52	1·54	1·56	1·58	1·60	1·62
3	1·16	1·19	1·21	1·24	1·26	1·29	1·31	1·33	1·35	1·36
4	·95	·98	1·01	1·04	1·07	1·09	1·11	1·14	1·16	1·18
5	·78	·82	·85	·88	·91	·93	·96	·98	1·00	1·03
6	·63	·67	·70	·73	·76	·79	·82	·85	·87	·89
7	·49	·53	·57	·60	·64	·67	·70	·73	·75	·78
8	·36	·41	·45	·48	·52	·55	·58	·61	·64	·67
9	·24	·29	·33	·37	·41	·44	·48	·51	·54	·57
10	·12	·17	·22	·26	·30	·34	·38	·41	·44	·47
11		·06	·11	·16	·20	·24	·28	·32	·35	·38
12				·05	·10	·14	·19	·22	·26	·29
13						·05	·09	·13	·17	·21
14								·04	·09	·12
15										·04

Tests of psychological preference and some other experimental data suffice to place a series of magnitudes in order of preference, without supplying metrical values. Analyses of variance, correlations, etc., can be carried out on such data by using the normal scores, appropriate to each position in order, in a sample of the size observed. Ties may be scored with the means of the ordinal values involved, but in such cases the sums of squares given in Table XXI will require correction.

Table XXI. Sums of Squares of Mean Deviations Tabulated

n	1	2	3	4	5	6	7	8	9	10	n
1		0·6272	1·4450	2·3018	3·1912	4·1250	5·0452	5·9646	6·9656	7·9320	10
11	8·8892	9·8662	10·8104	11·7846	12·8232	13·6600	14·7258	15·7454	16·6864	17·7144	20
21	18·6242	19·6862	20·6176	21·6040	22·6352	23·5470	24·5992	25·5808	26·5806	27·5454	30
31	28·5730	29·5960	30·5562	31·5152	32·5618	33·5166	34·5346	35·4840	36·4414	37·4288	40
41	38·4660	39·4654	40·4366	41·5788	42·5194	43·4922	44·4548	45·3626	46·5110	47·3830	50

TABLE XX. SCORES FOR ORDINAL (OR RANKED) DATA—*continued*

Ordinal number.	Size of Sample									
	31	32	33	34	35	36	37	38	39	40
1	2·06	2·07	2·08	2·09	2·11	2·12	2·13	2·14	2·15	2·16
2	1·63	1·65	1·66	1·68	1·69	1·70	1·72	1·73	1·74	1·75
3	1·38	1·40	1·42	1·43	1·45	1·46	1·48	1·49	1·50	1·52
4	1·20	1·22	1·23	1·25	1·27	1·28	1·30	1·32	1·33	1·34
5	1·05	1·07	1·09	1·11	1·12	1·14	1·16	1·17	1·19	1·20
6	·92	·94	·96	·98	1·00	1·02	1·03	1·05	1·07	1·08
7	·80	·82	·85	·87	·89	·91	·92	·94	·96	·98
8	·69	·72	·74	·76	·79	·81	·83	·85	·86	·88
9	·60	·62	·65	·67	·69	·72	·73	·75	·77	·79
10	·50	·53	·56	·58	·60	·63	·65	·67	·69	·71
11	·41	·44	·47	·50	·52	·54	·57	·59	·61	·63
12	·33	·36	·39	·41	·44	·47	·49	·51	·54	·56
13	·24	·28	·31	·34	·36	·39	·42	·44	·46	·49
14	·16	·20	·23	·26	·29	·32	·34	·37	·39	·42
15	·08	·12	·15	·18	·22	·24	·27	·30	·33	·35
16		·04	·08	·11	·14	·17	·20	·23	·26	·28
17				·04	·07	·10	·14	·16	·19	·22
18						·03	·07	·10	·13	·16
19								·03	·06	·09
20										·03

	41	42	43	44	45	46	47	48	49	50
1	2·17	2·18	2·19	2·20	2·21	2·22	2·22	2·23	2·24	2·25
2	1·76	1·78	1·79	1·80	1·81	1·82	1·83	1·84	1·85	1·85
3	1·53	1·54	1·55	1·57	1·58	1·59	1·60	1·61	1·62	1·63
4	1·36	1·37	1·38	1·40	1·41	1·42	1·43	1·44	1·45	1·46
5	1·22	1·23	1·25	1·26	1·27	1·28	1·30	1·31	1·32	1·33
6	1·10	1·11	1·13	1·14	1·16	1·17	1·18	1·19	1·21	1·22
7	·99	1·01	1·02	1·04	1·05	1·07	1·08	1·09	1·11	1·12
8	·90	·91	·93	·95	·96	·98	·99	1·00	1·02	1·03
9	·81	·83	·84	·86	·88	·89	·91	·92	·94	·95
10	·73	·75	·76	·78	·80	·81	·83	·84	·86	·87
11	·65	·67	·69	·71	·72	·74	·76	·77	·79	·80
12	·58	·60	·62	·64	·65	·67	·69	·70	·72	·74
13	·51	·53	·55	·57	·59	·60	·62	·64	·66	·67
14	·44	·46	·48	·50	·52	·54	·56	·58	·59	·61
15	·37	·40	·42	·44	·46	·48	·50	·52	·53	·55
16	·31	·33	·36	·38	·40	·42	·44	·46	·48	·49
17	·25	·27	·29	·32	·34	·36	·38	·40	·42	·44
18	·18	·21	·23	·26	·28	·30	·32	·34	·36	·38
19	·12	·15	·17	·20	·22	·25	·27	·29	·31	·33
20	·06	·09	·12	·14	·17	·19	·21	·24	·26	·28
21		·03	·06	·09	·11	·14	·16	·18	·21	·23
22				·03	·06	·08	·11	·13	·15	·18
23						·03	·05	·08	·10	·13
24								·03	·05	·08
25										·03

TABLE XXII. INITIAL DIFFERENCES OF POWERS OF NATURAL NUMBERS

These values, the so-called "differences of zero," enter into many combinatorial formulæ, as in Whitworth's *Choice and Chance*. They afford exact solutions of some important sampling problems (Stevens, (12), (13), 1937). In particular if s objects are distributed among n classes, the probabilities of any one object falling into each class being equal, the distribution of r, the number of occupied classes, is

$$\frac{1}{n^s} \frac{n!}{(n-r)!\,r!} \Delta^r 0^s$$

The values tabulated, $\Delta^r 0^s / r!$, are the rth differences of powers of natural numbers, advancing from 0^s, divided by $r!$, for values of s from 2 to 25.

r \ s	2	3	4	5	6	7	8
2	1	3	7	15	31	63	127
3		1	6	25	90	301	966
4			1	10	65	350	1701
5				1	15	140	1050
6					1	21	266
7						1	28
8							1

r \ s	9	10	11	12	13
2	255	511	1023	2047	4095
3	3025	9330	28501	86526	2 61625
4	7770	34105	1 45750	6 11501	25 32530
5	6951	42525	2 46730	13 79400	75 08501
6	2646	22827	1 79487	13 23652	93 21312
7	462	5880	63987	6 27396	57 15424
8	36	750	11880	1 59027	18 99612
9	1	45	1155	22275	3 59502
10		1	55	1705	39325
11			1	66	2431
12				1	78
13					1

r \ s	14	15	16	17	18
2	8191	16383	32767	65535	1 31071
3	7 88970	23 75101	71 41686	214 57825	644 39010
4	103 91745	423 55950	1717 98901	6943 37290	27988 06985
5	400 75035	2107 66920	10961 90550	56527 51651	2 89580 95545
6	634 36373	4206 93273	27349 26558	1 75057 49898	11 06872 51039
7	493 29280	4087 41333	32818 82604	2 57081 04786	19 74624 83400
8	209 12320	2166 27840	21417 64053	2 04159 95028	18 90360 65010
9	51 35130	671 28490	8207 84250	95288 22303	10 61753 95755
10	7 52752	126 62650	1937 54990	27583 34150	3 71121 63803
11	66066	14 79478	289 36908	5120 60978	83910 04908
12	3367	1 06470	27 57118	620 22324	12563 28866
13	91	4550	1 65620	49 10178	1258 54638
14	1	105	6020	2 49900	84 08778
15		1	120	7820	3 67200
16			1	136	9996
17				1	153
18					1

TABLE XXII. INITIAL DIFFERENCES OF POWERS OF NATURAL NUMBERS—*continued*

r \ s	19	20	21	22
2	2 62143	5 24287	10 48575	20 97151
3	1934 48101	5806 06446	17423 43625	52280 79450
4	1 12596 66950	4 52321 15901	18 15090 70050	72 77786 23825
5	14 75892 84710	74 92060 90500	379 12625 68401	1913 78219 12055
6	69 30816 01779	430 60788 95384	2658 56794 62804	16330 53393 45225
7	149 29246 34839	1114 35540 45652	8231 09572 14948	60276 23799 67440
8	170 97510 03480	1517 09326 62679	13251 10153 47084	1 14239 90799 91620
9	114 46146 26805	1201 12826 44725	12327 24764 65204	1 24196 33035 33920
10	47 72970 33785	591 75849 64655	7118 71322 91275	83514 37993 77954
11	12 94132 17791	190 08424 29486	2682 68516 89001	36628 25008 70286
12	2 34669 51330	41 10166 33391	683 30420 31078	10882 33560 61937
13	28924 39160	6 10686 60380	120 49092 18331	2249 68618 68481
14	2435 77530	63025 24580	14 93040 04500	329 51652 81331
15	139 16778	4523 29200	1 30874 62580	34 56159 43200
16	5 27136	223 50954	8099 44464	2 60465 74004
17	12597	7 41285	349 52799	14041 42047
18	171	15675	10 23435	533 74629
19	1	190	19285	13 89850
20		1	210	23485
21			1	231
22				1

r \ s	23	24	25
2	41 94303	83 88607	167 77215
3	1 56863 35501	4 70632 00806	14 11979 91025
4	291 63425 74750	1168 10566 34501	4677 12897 38810
5	9641 68881 84100	48500 07834 95250	2 43668 49741 10751
6	99896 98579 83405	6 09023 60360 84530	37 02641 70000 02430
7	4 38264 19991 17305	31 67746 38518 04540	227 83248 29987 16310
8	9 74195 50199 00400	82 31828 21583 20505	690 22372 11183 68580
9	12 32006 88117 96900	120 62257 43260 72500	1167 92145 10929 73005
10	9 59340 12973 13460	108 25408 17849 31500	1203 16339 21753 87500
11	4 86425 13089 51100	63 10016 56957 75560	802 35590 44384 62660
12	1 67216 27736 13530	24 93020 45923 13460	362 26262 08035 37080
13	40128 25603 52190	6 88883 60581 92000	114 48507 33488 09460
14	6862 91758 07115	1 36209 10216 51800	25 95811 03613 17200
15	847 94044 29331	19582 02422 47080	4 29939 46553 58000
16	76 23611 27264	2067 71824 65555	52665 51616 95960
17	4 99169 88803	161 09499 36915	4806 33313 93110
18	23648 85369	9 24849 25445	327 56785 94925
19	797 81779	38807 39170	16 62189 69675
20	18 59550	1169 72779	62201 94750
21	28336	24 54606	1685 19505
22	253	33902	32 00450
23	1	276	40250
24		1	300
25			1

TABLE XXIII. ORTHOGONAL POLYNOMIALS

(Values from $n' = 32$ to $n' = 75$ are due to V. Satakopan)

3

ξ'_1	ξ'_2
−1	+1
0	−2
+1	+1
2	6
1	3

4

ξ'_1	ξ'_2	ξ'_3
−3	+1	−1
−1	−1	+3
+1	−1	−3
+3	+1	+1
20	4	20
2	1	$\frac{10}{3}$

5

ξ'_1	ξ'_2	ξ'_3	ξ'_4
−2	+2	−1	+1
−1	−1	+2	−4
0	−2	0	+6
+1	−1	−2	−4
+2	+2	+1	+1
10	14	10	70
1	1	$\frac{5}{6}$	$\frac{35}{12}$

6

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−5	+5	−5	+1	−1
−3	−1	+7	−3	+5
−1	−4	+4	+2	−10
+1	−4	−4	+2	+10
+3	−1	−7	−3	−5
+5	+5	+5	+1	+1
70	84	180	28	252
2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{21}{10}$

7

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−3	+5	−1	+3	−1
−2	0	+1	−7	+4
−1	−3	+1	+1	−5
0	−4	0	+6	0
+1	−3	−1	+1	+5
+2	0	−1	−7	−4
+3	+5	+1	+3	+1
28	84	6	154	84
1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{20}$

8

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−7	+7	−7	+7	−7
−5	+1	+5	−13	+23
−3	−3	+7	−3	−17
−1	−5	+3	+9	−15
+1	−5	−3	+9	+15
+3	−3	−7	−3	+17
+5	+1	−5	−13	−23
+7	+7	+7	+7	+7
168	168	264	616	2184
2	1	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{7}{10}$

9

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−20	0	+18	0
+1	−17	−9	+9	+9
+2	−8	−13	−11	+4
+3	+7	−7	−21	−11
+4	+28	+14	+14	+4
60	2,772	990	2,002	468
1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{3}{20}$

10

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−4	−12	+18	+6
+3	−3	−31	+3	+11
+5	−1	−35	−17	+1
+7	+2	−14	−22	−14
+9	+6	+42	+18	+6
330	132	8,580	2,860	780
2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{5}{12}$	$\frac{1}{10}$

11

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−10	0	+6	0
+1	−9	−14	+4	+4
+2	−6	−23	−1	+4
+3	−1	−22	−6	−1
+4	+6	−6	−6	−6
+5	+15	+30	+6	+3
110	858	4,290	286	156
1	1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{40}$

12

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−35	−7	+28	+20
+3	−29	−19	+12	+44
+5	−17	−25	−13	+29
+7	+1	−21	−33	−21
+9	+25	−3	−27	−57
+11	+55	+33	+33	+33
572	12,012	5,148	8,008	15,912
2	3	$\frac{2}{3}$	$\frac{7}{24}$	$\frac{3}{20}$

13

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−14	0	+84	0
+1	−13	−4	+64	+20
+2	−10	−7	+11	+26
+3	−5	−8	−54	+11
+4	+2	−6	−96	−18
+5	+11	0	−66	−33
+6	+22	+11	+99	+22
182	2,002	572	68,068	6,188
1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{120}$

14

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−8	−24	+108	+60
+3	−7	−67	+63	+145
+5	−5	−95	−13	+139
+7	−2	−98	−92	+28
+9	+2	−66	−132	−132
+11	+7	+11	−77	−187
+13	+13	+143	+143	+143
910	728	97,240	136,136	235,144
2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{7}{30}$

15

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−56	0	+756	0
+1	−53	−27	+621	+675
+2	−44	−49	+251	+1000
+3	−29	−61	−249	+751
+4	−8	−58	−704	−44
+5	+19	−35	−869	−979
+6	+52	+13	−429	−1144
+7	+91	+91	+1001	+1001
280	37,128	39,780	6,466,460	10,581,480
1	3	$\frac{5}{6}$	$\frac{35}{12}$	$\frac{21}{20}$

16

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−21	−63	+189	+45
+3	−19	−179	+129	+115
+5	−15	−265	+23	+131
+7	−9	−301	−101	+77
+9	−1	−267	−201	−33
+11	+9	−143	−221	−143
+13	+21	+91	−91	−143
+15	+35	+455	+273	+143
1,360	5,712	1,007,760	470,288	201,552
2	1	$\frac{10}{2}$	$\frac{7}{12}$	$\frac{1}{10}$

17

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−24	0	+36	0
+1	−23	−7	+31	+55
+2	−20	−13	+17	+88
+3	−15	−17	−3	+83
+4	−8	−18	−24	+36
+5	+1	−15	−39	−39
+6	+12	−7	−39	−104
+7	+25	+7	−13	−91
+8	+40	+28	+52	+104
408	7,752	3,876	16,796	100,776
1	1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$

18

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−40	−8	+44	+220
+3	−37	−23	+33	+583
+5	−31	−35	+13	+733
+7	−22	−42	−12	+588
+9	−10	−42	−36	+156
+11	+5	−33	−51	−429
+13	+23	−13	−47	−871
+15	+44	+20	−12	−676
+17	+68	+68	+68	+884
1,938	23,256	23,256	28,424	6,953,544
2	$\frac{3}{2}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{3}{10}$

19				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−30	0	+396	0
+1	−29	−44	+352	+44
+2	−26	−83	+227	+74
+3	−21	−112	+42	+79
+4	−14	−126	−168	+54
+5	−5	−120	−354	+3
+6	+6	−89	−453	−58
+7	+19	−28	−388	−98
+8	+34	+68	−68	−68
+9	+51	+204	+612	+102
570		213,180		89,148
	13,566		2,288,132	
1	1	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{1}{40}$

20				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−33	−99	+1188	+396
+3	−31	−287	+948	+1076
+5	−27	−445	+503	+1441
+7	−21	−553	−77	+1351
+9	−13	−591	−687	+771
+11	−3	−539	−1187	−187
+13	+9	−377	−1402	−1222
+15	+23	−85	−1122	−1802
+17	+39	+357	−102	−1122
+19	+57	+969	+1938	+1938
2,660		4,903,140		31,201,800
	17,556		22,881,320	
2	1	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{7}{20}$

21				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−110	0	+594	0
+1	−107	−54	+540	+1404
+2	−98	−103	+385	+2444
+3	−83	−142	+150	+2819
+4	−62	−166	−130	+2354
+5	−35	−170	−406	+1063
+6	−2	−149	−615	−788
+7	+37	−98	−680	−2618
+8	+82	−12	−510	−3468
+9	+133	+114	0	−1938
+10	+190	+285	+969	+3876
770		432,630		121,687,020
	201,894		5,720,330	
1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{21}{40}$

22				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−20	−12	+702	+390
+3	−19	−35	+585	+1079
+5	−17	−55	+365	+1509
+7	−14	−70	+70	+1554
+9	−10	−78	−258	+1158
+11	−5	−77	−563	+363
+13	+1	−65	−775	−663
+15	+8	−40	−810	−1598
+17	+16	0	−570	−1938
+19	+25	+57	+57	−969
+21	+35	+133	+1197	+2261
3,542		96,140		40,562,340
	7,084		8,748,740	
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{7}{12}$	$\frac{7}{30}$

23				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−44	0	+858	0
+1	−43	−13	+793	+65
+2	−40	−25	+605	+116
+3	−35	−35	+315	+141
+4	−28	−42	−42	+132
+5	−19	−45	−417	+87
+6	−8	−43	−747	+12
+7	+5	−35	−955	−77
+8	+20	−20	−950	−152
+9	+37	+3	−627	−171
+10	+56	+35	+133	−76
+11	+77	+77	+1463	+209
1,012		32,890		340,860
	35,420		13,123,110	
1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{60}$

24				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−143	−143	+143	+715
+3	−137	−419	+123	+2005
+5	−125	−665	+85	+2893
+7	−107	−861	+33	+3171
+9	−83	−987	−27	+2721
+11	−53	−1023	−87	+1551
+13	−17	−949	−137	−169
+15	+25	−745	−165	−2071
+17	+73	−391	−157	−3553
+19	+127	+133	−97	−3743
+21	+187	+847	+33	−1463
+23	+253	+1771	+253	+4807
4,600		17,760,600		177,928,920
	394,680		394,680	
2	3	$\frac{10}{3}$	$\frac{1}{12}$	$\frac{3}{10}$

25				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−52	0	+858	0
+1	−51	−77	+803	+275
+2	−48	−149	+643	+500
+3	−43	−211	+393	+631
+4	−36	−258	+78	+636
+5	−27	−285	−267	+501
+6	−16	−287	−597	+236
+7	−3	−259	−857	−119
+8	+12	−196	−982	−488
+9	+29	−93	−897	−753
+10	+48	+55	−517	−748
+11	+69	+253	+253	−253
+12	+92	+506	+1518	+1012
1,300		1,480,050		7,803,900
	53,820		14,307,150	
1	1	$\frac{5}{6}$	$\frac{5}{12}$	$\frac{1}{20}$

26				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−28	−84	+1386	+330
+3	−27	−247	+1221	+935
+5	−25	−395	+905	+1381
+7	−22	−518	+466	+1582
+9	−18	−606	−54	+1482
+11	−13	−649	−599	+1067
+13	−7	−637	−1099	+377
+15	0	−560	−1470	−482
+17	+8	−408	−1614	−1326
+19	+17	−171	−1419	−1881
+21	+27	+161	−759	−1771
+23	+38	+598	+506	−506
+25	+50	+1150	+2530	+2530
5,850		7,803,900		48,384,180
	16,380		40,060,020	
2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{1}{10}$

27				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−182	0	+1638	0
+1	−179	−18	+1548	+3960
+2	−170	−35	+1285	+7304
+3	−155	−50	+870	+9479
+4	−134	−62	+338	+10058
+5	−107	−70	−262	+8803
+6	−74	−73	−867	+5728
+7	−35	−70	−1400	+1162
+8	+10	−60	−1770	−4188
+9	+61	−42	−1872	−9174
+10	+118	−15	−1587	−12144
+11	+181	+22	−782	−10879
+12	+250	+70	+690	−2530
+13	+325	+130	+2990	+16445
1,638		101,790		2,032,135,560
	712,530		56,448,210	
1	3	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{21}{40}$

28					29					30				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+1	−65	−39	+936	+1560	0	−70	0	+2184	0	+1	−112	−112	+12376	+1768
+3	−63	−115	+840	+4456	+1	−69	−104	+2080	+1768	+3	−109	−331	+11271	+5083
+5	−59	−185	+655	+6701	+2	−66	−203	+1775	+3298	+5	−103	−535	+9131	+7753
+7	−53	−245	+395	+7931	+3	−61	−292	+1290	+4373	+7	−94	−714	+6096	+9408
+9	−45	−291	+81	+7887	+4	−54	−366	+660	+4818	+9	−82	−858	+2376	+9768
+11	−35	−319	−259	+6457	+5	−45	−420	−66	+4521	+11	−67	−957	−1749	+8679
+13	−23	−325	−590	+3718	+6	−34	−449	−825	+3454	+13	−49	−1001	−5929	+6149
+15	−9	−305	−870	−22	+7	−21	−448	−1540	+1694	+15	−28	−980	−9744	+2384
+17	+7	−255	−1050	−4182	+8	−6	−412	−2120	−556	+17	−4	−884	−12704	−2176
+19	+25	−171	−1074	−7866	+9	+11	−336	−2460	−2946	+19	+23	−703	−14249	−6821
+21	+45	−49	−879	−9821	+10	+30	−215	−2441	−4958	+21	+53	−427	−13749	−10535
+23	+67	+115	−395	−8395	+11	+51	−44	−1930	−5885	+23	+86	−46	−10504	−11960
+25	+91	+325	+455	−1495	+12	+74	+182	−780	−4810	+25	+122	+450	−3744	−9360
+27	+117	+585	+1755	+13455	+13	+99	+468	+1170	−585	+27	+161	+1071	+7371	−585
					+14	+126	+819	+4095	+8190	+29	+203	+1827	+23751	+16965
7,308		2,103,660		1,354,757,040	2,030		4,207,320		500,671,080	8,990		21,360,240		2,145,733,200
	95,004		19,634,160			113,274		107,987,880			302,064		3,671,587,920	
2	1	$\frac{2}{3}$	$\frac{7}{24}$	$\frac{7}{20}$	1	1	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{7}{40}$	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{35}{12}$	$\frac{3}{10}$

31					32					33				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
+0	−80	0	+408	0	+1	−85	−51	+459	+255	0	−272	0	+3672	0
+1	−79	−119	+391	+221	+3	−83	−151	+423	+737	+1	−269	−27	+3537	+2565
+2	−76	−233	+341	+416	+5	−79	−245	+353	+1137	+2	−260	−53	+3139	+4864
+3	−71	−337	+261	+561	+7	−73	−329	+253	+1407	+3	−245	−77	+2499	+6649
+4	−64	−426	+156	+636	+9	−65	−399	+129	+1509	+4	−224	−98	+1652	+7708
+5	−55	−495	+33	+627	+11	−55	−451	−11	+1419	+5	−197	−115	+647	+7883
+6	−44	−539	−99	+528	+13	−43	−481	−157	+1131	+6	−164	−127	−453	+7088
+7	−31	−553	−229	+343	+15	−29	−485	−297	+661	+7	−125	−133	−1571	+5327
+8	−16	−532	−344	+88	+17	−13	−459	−417	+51	+8	−80	−132	−2616	+2712
+9	+1	−471	−429	−207	+19	+5	−399	−501	−627	+9	−29	−123	−3483	−519
+10	+20	−365	−467	−496	+21	+25	−301	−531	−1267	+10	+28	−105	−4053	−3984
+11	+41	−209	−439	−715	+23	+47	−161	−487	−1725	+11	+91	−77	−4193	−7139
+12	+64	+2	−324	−780	+25	+71	+25	−347	−1815	+12	+160	−38	−3756	−9260
+13	+89	+273	−99	−585	+27	+97	+261	−87	−1305	+13	+235	+13	−2581	−9425
+14	+116	+609	+261	0	+29	+125	+551	+319	+87	+14	+316	+77	−493	−6496
+15	+145	+1015	+783	+1131	+31	+155	+899	+899	+2697	+15	+403	+155	+2697	+899
										+16	+496	+248	+7192	+14384
2,480		6,724,520		9,536,592	10,912		5,379,616		54,285,216	2,992		417,384		1,547,128,656
	158,224		4,034,712			185,504		5,379,616			1,947,792		348,330,136	
1	1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{60}$	2	1	$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{30}$	1	3	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{3}{20}$

34				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−48	−144	+4104	+6840
3	−47	−427	+3819	+19855
5	−45	−695	+3263	+30917
7	−42	−938	+2464	+38864
9	−38	−1146	+1464	+42744
11	−33	−1309	+319	+41899
13	−27	−1417	−901	+36049
15	−20	−1460	−2112	+25376
17	−12	−1428	−3216	+10608
19	−3	−1311	−4101	−6897
21	+7	−1099	−4641	−25067
23	+18	−782	−4696	−41032
25	+30	−350	−4112	−51040
27	+43	+207	−2721	−50373
29	+57	+899	−341	−33263
31	+72	+1736	+3224	+7192
33	+88	+2728	+8184	+79112
13,090		51,477,360		46,929,569,232
	62,832		456,432,592	
2	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{7}{10}$

35				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−102	0	+23256	0
1	−101	−152	+22496	+3800
2	−98	−299	+20251	+7250
3	−93	−436	+16626	+10021
4	−86	−558	+11796	+11826
5	−77	−660	+6006	+12441
6	−66	−737	−429	+11726
7	−53	−784	−7124	+9646
8	−38	−796	−13624	+6292
9	−21	−768	−19404	+1902
10	−2	−695	−23869	−3118
11	+19	−572	−26354	−8173
12	+42	−394	−26124	−12458
13	+67	−156	−22374	−14937
14	+94	+147	−14229	−14322
15	+123	+520	−744	−9052
16	+154	+968	+19096	+2728
17	+187	+1496	+46376	+23188
3,570		15,775,320		4,045,652,520
	290,598		14,834,059,240	
1	1	$\frac{5}{6}$	$\frac{35}{12}$	$\frac{7}{40}$

36				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−323	−323	+2584	+12920
3	−317	−959	+2424	+37640
5	−305	−1565	+2111	+59063
7	−287	−2121	+1659	+75201
9	−263	−2607	+1089	+84381
11	−233	−3003	+429	+85371
13	−197	−3289	−286	+77506
15	−155	−3445	−1014	+60814
17	−107	−3451	−1706	+36142
19	−53	−3287	−2306	+5282
21	+7	−2933	−2751	−28903
23	+73	−2369	−2971	−62353
25	+145	−1575	−2889	−89685
27	+223	−531	−2421	−104067
29	+307	+783	−1476	−97092
31	+397	+2387	+44	−58652
33	+493	+4301	+2244	+23188
35	+595	+6545	+5236	+162316
15,540		307,618,740		199,046,103,984
	3,011,652		191,407,216	
2	3	$\frac{10}{3}$	$\frac{7}{24}$	$\frac{21}{20}$

37				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−114	0	+5814	0
1	−113	−34	+5644	+680
2	−110	−67	+5141	+1304
3	−105	−98	+4326	+1819
4	−98	−126	+3234	+2178
5	−89	−150	+1914	+2343
6	−78	−169	+429	+2288
7	−65	−182	−1144	+2002
8	−50	−188	−2714	+1492
9	−33	−186	−4176	+786
10	−14	−175	−5411	−64
11	+7	−154	−6286	−979
12	+30	−122	−6654	−1850
13	+55	−78	−6354	−2535
14	+82	−21	−5211	−2856
15	+111	+50	−3036	−2596
16	+142	+136	+374	−1496
17	+175	+238	+5236	+748
18	+210	+357	+11781	+4488
4,218		932,178		152,877,192
	383,838		980,961,982	
1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{40}$

38				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−60	−36	+918	+1530
3	−59	−107	+867	+4471
5	−57	−175	+767	+7061
7	−54	−238	+622	+9086
9	−50	−294	+438	+10362
11	−45	−341	+223	+10747
13	−39	−377	−13	+10153
15	−32	−400	−258	+8558
17	−24	−408	−498	+6018
19	−15	−399	−717	+2679
21	−5	−371	−897	−1211
23	+6	−322	−1018	−5290
25	+18	−250	−1058	−9070
27	+31	−153	−993	−11925
29	+45	−29	−797	−13079
31	+60	+124	−442	−11594
33	+76	+308	+102	−6358
35	+93	+525	+867	+3927
37	+111	+777	+1887	+20757
18,278		4,496,388		3,286,859,628
	109,668		25,479,532	
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{10}$

39				
ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−380	0	+1026	0
1	−377	−189	+999	+5049
2	−368	−373	+919	+9724
3	−353	−547	+789	+13669
4	−332	−706	+614	+16564
5	−305	−845	+401	+18143
6	−272	−959	+159	+18212
7	−233	−1043	−101	+16667
8	−188	−1092	−366	+13512
9	−137	−1101	−621	+8877
10	−80	−1065	−849	+3036
11	−17	−979	−1031	−3575
12	+52	−838	−1146	−10340
13	+127	−637	−1171	−16445
14	+208	−371	−1081	−20860
15	+295	−35	−849	−22321
16	+388	+376	−446	−19312
17	+487	+867	+159	−10047
18	+592	+1443	+999	+7548
19	+703	+2109	+2109	+35853
4,940		33,722,910		9,860,578,884
	4,496,388		32,224,114	
1	3	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{3}{20}$

40

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−133	−399	+39501	+627
3	−131	−1187	+37521	+1837
5	−127	−1945	+33631	+2917
7	−121	−2653	+27971	+3787
9	−113	−3291	+20751	+4377
11	−103	−3839	+12251	+4631
13	−91	−4277	+2821	+4511
15	−77	−4585	−7119	+4001
17	−61	−4743	−17079	+3111
19	−43	−4731	−26499	+1881
21	−23	−4529	−34749	+385
23	−1	−4117	−41129	−1265
25	+23	−3475	−44869	−2915
27	+49	−2583	−45129	−4365
29	+77	−1421	−40999	−5365
31	+107	+31	−31499	−5611
33	+139	+1793	−15579	−4741
35	+173	+3885	+7881	−2331
37	+209	+6327	+40071	+2109
39	+247	+9139	+82251	+9139

21,320 644,482,280 644,482,280
567,112 49,625,135,560
2 1 $\frac{10}{3}$ $\frac{35}{12}$ $\frac{1}{30}$

41

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−140	0	+8778	0
1	−139	−209	+8569	+4807
2	−136	−413	+7949	+9292
3	−131	−607	+6939	+13147
4	−124	−786	+5574	+16092
5	−115	−945	+3903	+17889
6	−104	−1079	+1989	+18356
7	−91	−1183	−91	+17381
8	−76	−1252	−2246	+14936
9	−59	−1281	−4371	+11091
10	−40	−1265	−6347	+6028
11	−19	−1199	−8041	+55
12	+4	−1078	−9306	−6380
13	+29	−897	−9981	−12675
14	+56	−651	−9891	−18060
15	+85	−335	−8847	−21583
16	+116	+56	−6646	−22096
17	+149	+527	−3071	−18241
18	+184	+1083	+2109	−8436
19	+221	+1729	+9139	+9139
20	+260	+2470	+18278	+36556

5,740 47,900,710 10,376,164,708
641,732 2,481,256,778
1 1 $\frac{5}{6}$ $\frac{7}{12}$ $\frac{7}{60}$

42

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−220	−44	+9614	+48070
3	−217	−131	+9177	+141151
5	−211	−215	+8317	+225181
7	−202	−294	+7062	+294546
9	−190	−366	+5454	+344262
11	−175	−429	+3549	+370227
13	−157	−481	+1417	+369473
15	−136	−520	−858	+340418
17	−112	−544	−3178	+283118
19	−85	−551	−5431	+199519
21	−55	−539	−7491	+93709
23	−22	−506	−9218	−27830
25	+14	−450	−10458	−155970
27	+53	−369	−11043	−278685
29	+95	−261	−10791	−380799
31	+140	−124	−9506	−443734
33	+188	+44	−6978	−445258
35	+239	+245	−2983	−359233
37	+293	+481	+2717	−155363
39	+350	+754	+10374	+201058
41	+410	+1066	+20254	+749398

24,682 9,075,924 4,389,117,671,484
1,629,012 3,084,805,724
2 $\frac{3}{2}$ $\frac{1}{3}$ $\frac{7}{12}$ $\frac{21}{10}$

43

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−154	0	+10626	0
1	−153	−46	+10396	+8740
2	−150	−91	+9713	+16948
3	−145	−134	+8598	+24113
4	−138	−174	+7086	+29766
5	−129	−210	+5226	+33501
6	−118	−241	+3081	+34996
7	−105	−266	+728	+34034
8	−90	−284	−1742	+30524
9	−73	−294	−4224	+24522
10	−54	−295	−6599	+16252
11	−33	−286	−8734	+6127
12	−10	−266	−10482	−5230
13	+15	−234	−11682	−16965
14	+42	−189	−12159	−27972
15	+71	−130	−11724	−36872
16	+102	−56	−10174	−41992
17	+135	+34	−7292	−41344
18	+170	+141	−2847	−32604
19	+207	+266	+3406	−13091
20	+246	+410	+11726	+20254
21	+287	+574	+22386	+70889

6,622 2,676,234 39,541,600,644
814,506 3,815,417,606
1 1 $\frac{1}{6}$ $\frac{7}{12}$ $\frac{7}{40}$

44

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
1	−161	−483	+5796	+1380
3	−159	−1439	+5556	+4060
5	−155	−2365	+5083	+6503
7	−149	−3241	+4391	+8561
9	−141	−4047	+3501	+10101
11	−131	−4763	+2441	+11011
13	−119	−5369	+1246	+11206
15	−105	−5845	−42	+10634
17	−89	−6171	−1374	+9282
19	−71	−6327	−2694	+7182
21	−51	−6293	−3939	+4417
23	−29	−6049	−5039	+1127
25	−5	−5575	−5917	−2485
27	+21	−4851	−6489	−6147
29	+49	−3857	−6664	−9512
31	+79	−2573	−6344	−12152
33	+111	−979	−5424	−13552
35	+145	+945	−3792	−13104
37	+181	+3219	−1329	−10101
39	+219	+5863	+2091	−3731
41	+259	+8897	+6601	+6929
43	+301	+12341	+12341	+22919

28,380 1,257,829,980 4,162,273,752
913,836 1,173,974,648
2 1 $\frac{10}{3}$ $\frac{7}{24}$ $\frac{1}{20}$

45

ξ'_1	ξ'_2	ξ'_3	ξ'_4	ξ'_5
0	−506	0	+9108	0
1	−503	−252	+8928	+4500
2	−494	−499	+8393	+8750
3	−479	−736	+7518	+12509
4	−458	−958	+6328	+15554
5	−431	−1160	+4858	+17689
6	−398	−1337	+3153	+18754
7	−359	−1484	+1268	+18634
8	−314	−1596	−732	+17268
9	−263	−1668	−2772	+14658
10	−206	−1695	−4767	+10878
11	−143	−1672	−6622	+6083
12	−74	−1594	−8232	+518
13	+1	−1456	−9482	−5473
14	+82	−1253	−10247	−11438
15	+169	−980	−10392	−16808
16	+262	−632	−9772	−20888
17	+361	−204	−8232	−22848
18	+466	+309	−5607	−21714
19	+577	+912	−1722	−16359
20	+694	+1610	+3608	−5494
21	+817	+2408	+10578	+12341
22	+946	+3311	+19393	+38786

7,590 92,036,340 12,006,558,900
9,203,634 2,934,936,620
1 3 $\frac{5}{6}$ $\frac{5}{12}$ $\frac{3}{10}$

Table XXIII. Tables of Orthogonal Functions for Values $n = 53$ to $n = 75$

46				47				48				49				50			
ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−88	−264	+1980	+3300	−184	0	+15180	0	−575	−115	+16445	+82225	−200	0	+17940	0	−104	−312	+96876	+10764
−87	−787	+1905	+9725	−183	−55	+14905	+3575	−569	−343	+15873	+242671	−199	−299	+17641	+9867	−103	−931	+93771	+31809
−85	−1295	+1757	+15631	−180	−109	+14087	+6968	−557	−565	+14743	+391231	−196	−593	+16751	+19272	−101	−1535	+87631	+51419
−82	−1778	+1540	+20692	−175	−161	+12747	+10003	−539	−777	+13083	+520401	−191	−877	+15291	+27767	−98	−2114	+78596	+68684
−78	−2226	+1260	+24612	−168	−210	+10920	+12516	−515	−975	+10935	+623307	−184	−1146	+13296	+34932	−94	−2658	+66876	+82764
−73	−2629	+925	+27137	−159	−255	+8655	+14361	−485	−1155	+8355	+693957	−175	−1395	+10815	+40389	−89	−3157	+52751	+92917
−67	−2977	+545	+28067	−148	−295	+6015	+15416	−449	−1313	+5413	+727493	−164	−1619	+7911	+43816	−83	−3601	+36571	+98527
−60	−3260	+132	+27268	−135	−329	+3077	+15589	−407	−1445	+2193	+720443	−151	−1813	+4661	+44961	−76	−3980	+18756	+99132
−52	−3468	−300	+24684	−120	−356	−68	+14824	−359	−1547	−1207	+670973	−136	−1972	+1156	+43656	−68	−4284	−204	+94452
−43	−3591	−735	+20349	−103	−375	−3315	+13107	−305	−1615	−4675	+579139	−119	−2091	−2499	+39831	−59	−4503	−19749	+84417
−33	−3619	−1155	+14399	−84	−385	−6545	+10472	−245	−1645	−8085	+447139	−100	−2165	−6185	+33528	−49	−4627	−39249	+69195
−22	−3542	−1540	+7084	−63	−385	−9625	+7007	−179	−1633	−11297	+279565	−79	−2189	−9769	+24915	−38	−4646	−58004	+49220
−10	−3350	−1868	−1220	−40	−374	−12408	+2860	−107	−1575	−14157	+83655	−56	−2158	−13104	+14300	−26	−4550	−75244	+25220
+3	−3033	−2115	−9999	−15	−351	−14733	−1755	−29	−1467	−16497	−130455	−31	−2067	−16029	+2145	−13	−4329	−90129	−1755
+17	−2581	−2255	−18589	+12	−315	−16425	−6552	+55	−1305	−18135	−349479	−4	−1911	−18369	−10920	+1	−3973	−101749	−30305
+32	−1984	−2260	−26164	+41	−265	−17295	−11167	+145	−1085	−18875	−556729	+25	−1685	−19935	−24083	+16	−3472	−109124	−58652
+48	−1232	−2100	−31724	+72	−200	−17140	−15152	+241	−803	−18507	−731863	+56	−1384	−20524	−36336	+32	−2816	−111204	−84612
+65	−315	−1743	−34083	+105	−119	−15743	−17969	+343	−455	−16807	−850633	+89	−1003	−19919	−46461	+49	−1995	−106869	−105567
+83	+777	−1155	−31857	+140	−21	−12873	−18984	+451	−37	−13537	−884633	+124	−537	−17889	−53016	+67	−999	−94929	−118437
+102	+2054	−300	−23452	+177	+95	−8285	−17461	+565	+455	−8445	−801047	+161	+19	−14189	−54321	+86	+182	−74124	−119652
+122	+3526	+860	−7052	+216	+230	−1720	−12556	+685	+1025	−1265	−562397	+200	+670	−8560	−48444	+106	+1558	−43124	−105124
+143	+5203	+2365	+19393	+257	+385	+7095	−3311	+811	+1677	+8283	−126291	+241	+1421	−729	−33187	+127	+3139	−529	−70219
+165	+7095	+4257	+58179	+300	+561	+18447	+11352	+943	+2415	+20493	+554829	+284	+2277	+9591	−6072	+149	+4935	+55131	−9729
				+345	+759	+32637	+32637	+1081	+3243	+35673	+1533939	+329	+3243	+22701	+35673	+172	+6956	+125396	+82156
												+376	+4324	+38916	+95128	+196	+9212	+211876	+211876
	429,502,920		27,214,866,840		4,994,220		8,629,104,120		92,620,080		19,208,385,771,120		167,230,700		74,451,107,640		770,715,400		372,255,538,200
285,384		143,167,640		1,271,256		8,518,474,580		12,712,560		10,301,411,120		1,566,040		12,408,517,940		433,160		372,255,538,200	
$\frac{1}{2}$	$\frac{5}{3}$	$\frac{1}{12}$	$\frac{1}{10}$	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{1}{20}$	3	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{21}{10}$	1	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{7}{60}$	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{35}{12}$	$\frac{7}{30}$

From this point onwards ξ'_1 is omitted.

TABLE XXIII. TABLES OF ORTHOGONAL FUNCTIONS FOR VALUES $n = 53$ TO $n = 75$—*continued*

51				52				53				54				55			
ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−650	0	+21060	0	−225	−135	+1620	+2700	−234	0	+3510	0	−364	−364	+26390	+18850	−252	0	+142506	0
−647	−324	+20736	+7452	−223	−403	+1572	+7988	−233	−70	+3460	+2900	−361	−1087	+25665	+55825	−251	−377	+140621	+47125
−638	−643	+19771	+14582	−219	−665	+1477	+12943	−230	−139	+3311	+5684	−355	−1795	+24229	+90643	−248	−749	+135001	+92500
−623	−952	+18186	+21077	−213	−917	+1337	+17353	−225	−206	+3066	+8239	−346	−2478	+22110	+121926	−243	−1111	+125751	+134417
−602	−1246	+16016	+26642	−205	−1155	+1155	+21021	−218	−270	+2730	+10458	−334	−3126	+19350	+148386	−236	−1458	+113046	+171252
−575	−1520	+13310	+31009	−195	−1375	+935	+23771	−209	−330	+2310	+12243	−319	−3729	+16005	+168861	−227	−1785	+97131	+201507
−542	−1769	+10131	+33946	−183	−1573	+682	+25454	−198	−385	+1815	+13508	−301	−4277	+12145	+182351	−216	−2087	+78321	+223852
−503	−1988	+6556	+35266	−169	−1745	+402	+25954	−185	−434	+1256	+14182	−280	−4760	+7854	+188054	−203	−2359	+57001	+237167
−458	−2172	+2676	+34836	−153	−1887	+102	+25194	−170	−476	+646	+14212	−256	−5168	+3230	+185402	−188	−2596	+33626	+240584
−407	−2316	−1404	+32586	−135	−1995	−210	+23142	−153	−510	0	+13566	−229	−5491	−1615	+174097	−171	−2793	+8721	+233529
−350	−2415	−5565	+28518	−115	−2065	−525	+19817	−134	−535	−665	+12236	−199	−5719	−6555	+154147	−152	−2945	−17119	+215764
−287	−2464	−9674	+22715	−93	−2093	−833	+15295	−113	−550	−1330	+10241	−166	−5842	−11450	+125902	−131	−3047	−43229	+187429
−218	−2458	−13584	+15350	−69	−2075	−1123	+9715	−90	−554	−1974	+7630	−130	−5850	−16146	+90090	−108	−3094	−68874	+149084
−143	−2392	−17134	+6695	−43	−2007	−1383	+3285	−65	−546	−2574	+4485	−91	−5733	−20475	+47853	−83	−3081	−93249	+101751
−62	−2261	−20149	−2870	−15	−1885	−1600	−3712	−38	−525	−3105	+924	−49	−5481	−24255	+783	−56	−3003	−115479	+46956
+25	−2060	−22440	−12848	+15	−1705	−1760	−10912	−9	−490	−3540	−2896	−4	−5084	−27290	−49042	−27	−2855	−134619	−13229
+118	−1784	−23804	−22616	+47	−1463	−1848	−17864	+22	−440	−3850	−6776	+44	−4532	−29370	−99022	+4	−2632	−149654	−76144
+217	−1428	−24024	−31416	+81	−1155	−1848	−24024	+55	−374	−4004	−10472	+95	−3815	−30271	−145999	+37	−2329	−159499	−138499
+322	−987	−22869	−38346	+117	−777	−1743	−28749	+90	−291	−3969	−13692	+149	−2923	−29755	−186221	+72	−1941	−162999	−196332
+433	−456	−20094	−42351	+155	−325	−1515	−31291	+127	−190	−3710	−16093	+206	−1846	−27570	−215306	+109	−1463	−158929	−244967
+550	+170	−15440	−42214	+195	+205	−1145	−30791	+166	−70	−3190	−17278	+266	−574	−23450	−228206	+148	−890	−145994	−278972
+673	+896	−8634	−36547	+237	+817	−613	−26273	+207	+70	−2370	−16793	+329	+903	−17115	−219171	+189	−217	−122829	−292117
+802	+1727	+611	−23782	+281	+1515	+102	−16638	+250	+231	−1209	−14124	+395	+2595	−8271	−181713	+232	+561	−87999	−277332
+937	+2668	+12596	−2162	+327	+2303	+1022	−658	+295	+414	+336	−8694	+464	+4512	+3390	−108570	+277	+1449	−39999	−226665
+1078	+3724	+27636	+30268	+375	+3185	+2170	+23030	+342	+620	+2310	+140	+536	+6664	+18190	+8330	+324	+2452	+22746	−131240
+1225	+4900	+46060	+75670	+425	+4165	+3570	+55930	+391	+850	+4760	+13090	+611	+9061	+36465	+177905	+373	+3575	+101881	+18785
								+442	+1105	+7735	+30940	+689	+11713	+58565	+409955	+424	+4823	+199121	+234260
																+477	+6201	+316251	+527085
221,375,700		47,861,426,340		162,342,180		26,358,466,680		11,595,870		8,131,867,380		1,321,929,180		1,439,340,526,260		375,842,610		2,398,900,877,100	
17,218,110		17,803,525,740		2,108,340		108,228,120		2,319,174		514,083,570		5,729,724		29,816,847,060		2,791,404		879,596,988,270	
3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{3}{40}$	1	$\frac{2}{3}$	$\frac{1}{24}$	$\frac{1}{20}$	1	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{40}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{3}{10}$	1	$\frac{5}{6}$	$\frac{35}{12}$	$\frac{7}{20}$

56				57				58				59				60			
ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−261	−783	+30537	+50895	−812	0	+32886	0	−140	−84	+35154	+2790	−290	0	+5394	0	−899	−899	+14384	+14384
−259	−2339	+29757	+150865	−809	−81	+32481	+163215	−139	−251	+34317	+8277	−289	−434	+5332	+1488	−893	−2687	+14064	+42704
−255	−3865	+28211	+245417	−800	−161	+31273	+320788	−137	−415	+32657	+13487	−286	−863	+5147	+2928	−881	−4445	+13429	+69689
−249	−5341	+25927	+331079	−785	−239	+29283	+467203	−134	−574	+30202	+18242	−281	−1282	+4842	+4273	−863	−6153	+12489	+94479
−241	−6747	+22947	+404589	−764	−314	+26546	+597196	−130	−726	+26994	+22374	−274	−1686	+4422	+5478	−839	−7791	+11259	+116259
−231	−8063	+19327	+462979	−737	−385	+23111	+705881	−125	−869	+23089	+25729	−265	−2070	+3894	+6501	−809	−9339	+9759	+134277
−219	−9269	+15137	+503659	−704	−451	+19041	+788876	−119	−1001	+18557	+28171	−254	−2429	+3267	+7304	−773	−10777	+8014	+147862
−205	−10345	+10461	+524501	−665	−511	+14413	+842429	−112	−1120	+13482	+29586	−241	−2758	+2552	+7854	−731	−12085	+6054	+156442
−189	−11271	+5397	+523923	−620	−564	+9318	+863544	−104	−1224	+7962	+29886	−226	−3052	+1762	+8124	−683	−13243	+3914	+159562
−171	−12027	+57	+500973	−569	−609	+3861	+850107	−95	−1311	+2109	+29013	−209	−3306	+912	+8094	−629	−14231	+1634	+156902
−151	−12593	−5433	+455413	−512	−645	−1839	+801012	−85	−1379	−3951	+26943	−190	−3515	+19	+7752	−569	−15029	−741	+148295
−129	−12949	−10933	+387803	−449	−671	−7649	+716287	−74	−1426	−10078	+23690	−169	−3674	−898	+7095	−503	−15617	−3161	+133745
−105	−13075	−16289	+299585	−380	−686	−13422	+597220	−62	−1450	−16118	+19310	−146	−3778	−1818	+6130	−431	−15975	−5571	+113445
−79	−12951	−21333	+193167	−305	−689	−18997	+446485	−49	−1449	−21903	+13905	−121	−3822	−2718	+4875	−353	−16083	−7911	+87795
−51	−12557	−25883	+72007	−224	−679	−24199	+268268	−35	−1421	−27251	+7627	−94	−3801	−3573	+3360	−269	−15921	−10116	+57420
−21	−11873	−29743	−59303	−137	−655	−28839	+68393	−20	−1364	−31966	+682	−65	−3710	−4356	+1628	−179	−15469	−12116	+23188
+11	−10879	−32703	−194953	−44	−616	−32714	−145552	−4	−1276	−35838	−6666	−34	−3544	−5038	−264	−83	−14707	−13836	−13772
+45	−9555	−34539	−327831	+55	−561	−35607	−364089	+13	−1155	−38643	−14091	−1	−3298	−5588	2244	+19	−13615	−15196	−52052
+81	−7881	−35013	−449439	+160	−489	−37287	−575724	+31	−999	−40143	−21201	+34	−2967	−5973	−4224	+127	−12173	−16111	−89947
+119	−5837	−33873	−549809	+271	−399	−37509	−766821	+50	−806	−40086	−27534	+71	−2546	−6158	−6099	+241	−10361	−16491	−125437
+159	−3403	−30853	−617419	+388	−290	−36014	−921476	+70	−574	−38206	−32554	+110	−2030	−6106	−7746	+361	−8159	−16241	−156169
+201	−559	−25673	−639109	+511	−161	−32529	−1021391	+91	−301	−34223	−35647	+151	−1414	−5778	−9023	+487	−5547	−15261	−179439
+245	+2715	−18039	−599997	+640	−11	−26767	−1045748	+113	+15	−27843	−36117	+194	−693	−5133	−9768	+619	−2505	−13446	−192174
+291	+6439	−7643	−483395	+775	+161	−18427	−971083	+136	+376	−18758	−33182	+239	+138	−4128	−9798	+757	+987	−10686	−190914
+339	+10633	+5837	−270725	+916	+356	−7194	−771160	+160	+784	−6646	−25970	+286	+1084	−2718	−8908	+901	+4949	−6866	−171794
+389	+15317	+22737	+58565	+1063	+575	+7261	−416845	+185	+1241	+8829	−13515	+335	+2150	−856	−6870	+1051	+9401	−1866	−130526
+441	+20511	+43407	+527085	+1216	+819	+25281	+124020	+211	+1749	+28017	+5247	+386	+3341	+1507	−3432	+1207	+14363	+4439	−62381
+495	+26235	+68211	+1159587	+1375	+1089	+47223	+886743	+238	+2310	+51282	+31482	+439	+4662	+4422	+1683	+1369	+19855	+12179	+37829
				+1540	+1386	+73458	+1909908	+266	+2926	+79002	+66462	+494	+6118	+7942	+8778	+1537	+25897	+21489	+175769
												+551	+7714	+12122	+18183	+1711	+32509	+32509	+357599
	6,822,988,920		11,706,636,280,248		19,310,346		32,021,093,354,796		87,254,156		39,097,794,084		614,745,190		2,950,776,912		11,065,413,420		1,150,802,995,680
	3,054,744		41,392,799,448		30,038,316		48,556,937,814		910,252		56,802,455,556		3,966,098		1,352,439,418		38,826,012		9,835,923,040
	1	$\frac{10}{3}$	$\frac{7}{12}$	$\frac{17}{10}$	3	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{21}{20}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{7}{12}$	$\frac{1}{30}$	1	$\frac{5}{6}$	$\frac{1}{12}$	$\frac{1}{120}$	3	$\frac{10}{3}$	$\frac{5}{24}$	$\frac{3}{20}$

TABLE XXIII. TABLES OF ORTHOGONAL FUNCTIONS FOR VALUES $n = 53$ TO $n = 75$—*continued*

61				62				63				64				65			
ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ_2'	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−310	0	+43152	0	−160	−96	+45936	+76560	−992	0	+49104	0	−341	−1023	+52173	+86955	−352	0	+278256	0
−309	−464	+42688	+5104	−159	−287	+44979	+227447	−989	−99	+48609	+244035	−339	−3059	+51153	+258485	−351	−527	+275621	+13175
−306	−923	+41303	+10054	−157	−475	+43079	+371677	−980	−197	+47131	+481168	−335	−5065	+49127	+422917	−348	−1049	+267751	+26000
−301	−1372	+39018	+14699	−154	−658	+40264	+504952	−965	−293	+44691	+704623	−329	−7021	+46123	+575659	−343	−1561	+254751	+38131
−294	−1806	+35868	+18894	−150	−834	+36576	+623184	−944	−386	+41324	+907876	−321	−8907	+42183	+712329	−336	−2058	+236796	+49236
−285	−2220	+31902	+22503	−145	−1001	+32071	+722579	−917	−475	+37079	+1084781	−311	−10703	+37363	+828839	−327	−2535	+214131	+59001
−274	−2609	+27183	+25402	−139	−1157	+26819	+799721	−884	−559	+32019	+1229696	−299	−12389	+31733	+921479	−316	−2987	+187071	+67136
−261	−2968	+21788	+27482	−132	−1300	+20904	+851656	−845	−637	+26221	+1337609	−285	−13945	+25377	+987001	−303	−3409	+156001	+73381
−246	−3292	+15808	+28652	−124	−1428	+14424	+875976	−800	−708	+19776	+1404264	−269	−15351	+18393	+1022703	−288	−3796	+121376	+77512
−229	−3576	+9348	+28842	−115	−1539	+7491	+870903	−749	−771	+12789	+1426287	−251	−16587	+10893	+1026513	−271	−4143	+83721	+79347
−210	−3815	+2527	+28006	−105	−1631	+231	+835373	−692	−825	+5379	+1401312	−231	−17633	+3003	+997073	−252	−4445	+43631	+78752
−189	−4004	−4522	+26125	−94	−1702	−7216	+769120	−629	−869	−2321	+1328107	−209	−18469	−5137	+933823	−231	−4697	+1771	+75647
−166	−4138	−11652	+23210	−82	−1750	−14696	+672760	−560	−902	−10164	+1206700	−185	−19075	−13373	+837085	−208	−4894	−41124	+70012
−141	−4212	−18702	+19305	−69	−1773	−22041	+547875	−485	−923	−17989	+1038505	−159	−19431	−21537	+708147	−183	−5031	−84249	+61893
−114	−4221	−25497	+14490	−55	−1769	−29069	+397097	−404	−931	−25621	+826448	−131	−19517	−29447	+549347	−156	−5103	−126729	+51408
−85	−4160	−31848	+8884	−40	−1736	−35584	+224192	−317	−925	−32871	+575093	−101	−19313	−36907	+364157	−127	−5105	−167619	+38753
−54	−4024	−37552	+2648	−24	−1672	−41376	+34144	−224	−904	−39536	+290768	−69	−18799	−43707	+157267	−96	−5032	−205904	+24208
−21	−3808	−42392	−4012	−7	−1575	−46221	−166761	−125	−867	−45399	−18309	−35	−17955	−49623	−6533I	−63	−4879	−240499	+8143
+14	−3507	−46137	−10842	+11	−1443	−49881	−370851	− 20	−813	−50229	−341904	+1	−16761	−54417	−296259	−28	−4641	−270249	−8976
+51	−3116	−48542	−17537	+30	−1274	−52104	−568984	+91	−741	−53781	−667641	+39	−15197	−57837	−526669	+9	−4313	−293929	−26581
+90	−2630	−49348	−23738	+50	−1066	−52624	−750464	+208	−650	−55796	−980876	+79	−13243	−59617	−746159	+48	−3890	−310244	−43996
+131	−2044	−48282	−29029	+71	−817	−51161	−902957	+331	−539	−56001	−1264571	+121	−10879	−59477	−942689	+89	−3367	−317829	−60431
+174	−1353	−45057	−32934	+93	−525	−47421	−1012407	+460	−407	−54109	−1499168	+165	−8085	−57123	−1102497	+132	−2739	−315249	−74976
+219	−552	−39372	−34914	+116	−188	−41096	−1062952	+595	−253	−49819	−1662463	+211	−4841	−52247	−1210015	+177	−2001	−300999	−86595
+266	+364	−30912	−34364	+140	+196	−31864	−1036840	+736	−76	−42816	−1729480	+259	−1127	−44527	−1247785	+224	−1148	−273504	−94120
+315	+1400	−19348	−30610	+165	+629	−19389	−914345	+883	+125	−32771	−1672345	+309	+3077	−33627	−1196375	+273	−175	−231119	−96245
+366	+2561	−4337	−22906	+191	+1113	−3321	−673683	+1036	+351	−19341	−1460160	+361	+7791	−19197	−1034295	+324	+923	−172129	−91520
+419	+3852	+14478	−10431	+218	+1650	+16704	−290928	+1195	+603	−2169	−1058877	+415	+13035	−873	−737913	+377	+2151	−94749	−78345
+474	+5278	+37468	+7714	+246	+2242	+41064	+260072	+1360	+882	+19116	−431172	+471	+18829	+21723	−281371	+432	+3514	+2876	−54964
+531	+6844	+65018	+32509	+275	+2891	+70151	+1007779	+1531	+1189	+44899	+463681	+529	+25193	+48983	+363499	+489	+5017	+122671	−19459
+590	+8555	+97527	+65018	+305	+3599	+104371	+1983049	+1708	+1525	+75579	+1669936	+589	+32147	+81313	+1227259	+548	+6665	+266631	+30256
								+1891	+1891	+111569	+3235501	+651	+39711	+119133	+2342949	+609	+8463	+436821	+96441
																+672	+10416	+635376	+181536
	776,520,240		38,360,099,856		139,238,112		35,981,773,664,928		38,939,472		96,582,655,626,912		17,392,964,160		51,066,691,480,896		1,211,804,880		309,120,408,480
4,685,898		89,506,899,664		1,270,752		103,639,568,032		49,559,328		119,721,569,968		5,957,952		137,984,182,336		6,438,432		3,967,045,242,160	

66				67				68				69				70			
ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5
−544	−544	+8432	+42160	−374	0	+8976	0	−385	−231	+33264	+2640	−1190	0	+70686	0	−204	−612	+373626	+124542
−541	−1627	+8277	+125395	−373	−112	+8896	+2480	−383	−691	+32688	+7856	−1187	−594	+70092	+175824	−203	−1831	+367521	+370777
−535	−2695	+7969	+205393	−370	−223	+8657	+4898	−379	−1145	+31543	+12881	−1178	−1183	+68317	+347504	−201	−3035	+355381	+608507
−526	−3738	+7512	+280056	−365	−332	+8262	+7193	−373	−1589	+29843	+17591	−1163	−1762	+65382	+510959	−198	−4214	+337346	+832202
−514	−4746	+6912	+347376	−358	−438	+7716	+9306	−365	−2019	+27609	+21867	−1142	−2326	+61322	+662234	−194	−5358	+313626	+1036542
−499	−5709	+6177	+405471	−349	−540	+7026	+11181	−355	−2431	+24869	+25597	−1115	−2870	+56186	+797563	−189	−6457	+284501	+1216501
−481	−6617	+5317	+452621	−338	−637	+6201	+12766	−343	−2821	+21658	+28678	−1082	−3389	+50037	+913432	−183	−7501	+250321	+1367431
−460	−7460	+4344	+487304	−325	−728	+5252	+14014	−329	−3185	+18018	+31018	−1043	−3878	+42952	+1006642	−176	−8480	+211506	+1485146
−436	−8228	+3272	+508232	−310	−812	+4192	+14884	−313	−3519	+13998	+32538	−998	−4332	+35022	+1074372	−168	−9384	+168546	+1566006
−409	−8911	+2117	+514387	−293	−888	+3036	+15342	−295	−3819	+9654	+33174	−947	−4746	+26352	+1114242	−159	−10203	+122001	+1607001
−379	−9499	+897	+505057	−274	−955	+1801	+15362	−275	−4081	+5049	+32879	−890	−5115	+17061	+1124376	−149	−10927	+72501	+1605835
−346	−9982	−368	+479872	−253	−1012	+506	+14927	−253	−4301	+253	+31625	−827	−5434	+7282	+1103465	−138	−11546	+20746	+1561010
−310	−10350	−1656	+438840	−230	−1058	−828	+14030	−229	−4475	−4657	+29405	−758	−5698	−2838	+1050830	−126	−12050	−32494	+1471910
−271	−10593	−2943	+382383	−205	−1092	−2178	+12675	−203	−4599	−9597	+26235	−683	−5902	−13138	+966485	−113	−12429	−86379	+1338885
−229	−10701	−4203	+311373	−178	−1113	−3519	+10878	−175	−4669	−14476	+22156	−602	−6041	−23443	+851200	−99	−12673	−139999	+1163335
−184	−10664	−5408	+227168	−149	−1120	−4824	+8668	−145	−4681	−19196	+17236	−515	−6110	−33564	+706564	−84	−12772	−192374	+947794
−136	−10472	−6528	+131648	−118	−1112	−6064	+6088	−113	−4631	−23652	+11572	−422	−6104	−43298	+535048	−68	−12716	−242454	+696014
−85	−10115	−7531	+27251	−85	−1088	−7208	+3196	−79	−4515	−27732	+5292	−323	−6018	−52428	+340068	−51	−12495	−289119	+413049
−31	−9583	−8383	−82991	−50	−1047	−8223	+66	−43	−4329	−31317	−1443	−218	−5847	−60723	+126048	−33	−12099	−331179	+105339
+26	−8866	−9048	−195416	−13	−988	−9074	−3211	−5	−4069	−34281	−8437	−107	−5586	−67938	−101517	−14	−11518	−367374	−219206
+86	−7954	−9488	−305696	+26	−910	−9724	−6526	+35	−3731	−36491	−15457	+10	−5230	−73814	−335998	+6	−10742	−396374	−551122
+149	−6837	−9663	−408801	+67	−812	−10134	−9751	+77	−3311	−37807	−22231	+133	−4774	−78078	−569569	+27	−9761	−416779	−879307
+215	−5505	−9531	−498963	+110	−693	−10263	−12738	+121	−2805	−38082	−28446	+262	−4213	−80443	−793144	+49	−8565	−427119	−1190937
+284	−3948	−9048	−569640	+155	−552	−10068	−15318	+167	−2209	−37162	−33746	+397	−3542	−80608	−996314	+72	−7144	−425854	−1471382
+356	−2156	−8168	−613480	+202	−388	−9504	−17300	+215	−1519	−34886	−37730	+538	−2756	−78258	−1167284	+96	−5488	−411374	−1704122
+431	−119	−6843	−622285	+251	−200	−8524	−18470	+265	−731	−31086	−39950	+685	−1850	−73064	−1292810	+121	−3587	−381999	−1870663
+509	+2173	−5023	−586975	+302	+13	−7079	−18590	+317	+159	−25587	−39909	+838	−819	−64683	−1358136	+147	−1431	−335979	−1950453
+590	+4730	−2656	−497552	+355	+252	−5118	−17397	+371	+1155	−18207	−37059	+997	+342	−52758	−1346931	+174	+990	−271494	−1920798
+674	+7562	+312	−343064	+410	+518	−2588	−14602	+427	+2261	−8757	−30799	+1162	+1638	−36918	−1241226	+202	+3686	−186654	−1756778
+761	+10679	+3937	−111569	+467	+812	+566	−9889	+485	+3481	+2959	−20473	+1333	+3074	−16778	−1021351	+231	+6667	−79499	−1431163
+851	+14091	+8277	+209901	+526	+1135	+4401	−2914	+545	+4819	+17144	−5368	+1510	+4655	+8061	−665872	+261	+9943	+52001	−914329
+944	+17808	+13392	+635376	+587	+1488	+8976	+6696	+607	+6279	+34008	+15288	+1693	+6386	+38012	−151528	+292	+13524	+209946	−174174
+1040	+21840	+19344	+1179984	+650	+1872	+14352	+19344	+671	+7865	+53768	+42328	+1882	+8272	+73502	+546832	+324	+17420	+396506	+823966
				+715	+2288	+20592	+35464	+737	+9581	+76648	+76648	+2077	+10318	+114972	+1456312	+357	+21641	+613921	+2117401
												+2278	+12529	+162877	+2606032	+391	+26197	+864501	+3746171
15,636,192		3,716,201,632			59,938,736		11,993,196,176		1,063,912,564		56,484,085,216		1,841,387,130		65,832,201,319,248		8,146,743,060		137,150,419,415,100
	5,394,486,240		13,168,529,401,248	7,492,342		4,255,650,256		8,068,676		59,579,103,584		78,119,454		271,829,660,102		2,331,924		7,736,690,325,980	
$\frac{3}{2}$	$\frac{5}{3}$	$\frac{1}{12}$	$\frac{3}{10}$	1	$\frac{1}{6}$	$\frac{71}{12}$	$\frac{1}{120}$	1	$\frac{8}{3}$	$\frac{7}{24}$	$\frac{1}{60}$	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{21}{40}$	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{35}{12}$	$\frac{7}{10}$

71				72				73				74				75			
ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5	ξ'_2	ξ'_3	ξ'_4	ξ'_5
-420	0	+79254	0	-1295	-259	+83657	+59755	-444	0	+12654	0	-228	-684	+13338	+22230	-1406	0	+493506	0
-419	-629	+78625	+131461	-1289	-775	+82365	+177973	-443	-133	+12559	+20995	-227	-2047	+13143	+66235	-1403	-702	+489996	+35100
-416	-1253	+76745	+259996	-1277	-1285	+79795	+292333	-440	-265	+12275	+41548	-225	-3395	+12755	+108881	-1394	-1399	+479501	+69500
-411	-1867	+73635	+382721	-1259	-1785	+75975	+400323	-435	-395	+11805	+61223	-222	-4718	+12178	+149282	-1379	-2086	+462126	+102509
-404	-2466	+69330	+496836	-1235	-2271	+70947	+499521	-428	-522	+11154	+79596	-218	-6006	+11418	+186582	-1358	-2758	+438046	+133454
-395	-3045	+63879	+599667	-1205	-2739	+64767	+587631	-419	-645	+10329	+96261	-213	-7249	+10483	+219967	-1331	-3410	+407506	+161689
-384	-3599	+57345	+688708	-1169	-3185	+57505	+662519	-408	-763	+9339	+110836	-207	-8437	+9383	+248677	-1298	-4037	+370821	+186604
-371	-4123	+49805	+761663	-1127	-3605	+49245	+722249	-395	-875	+8195	+122969	-200	-9560	+8130	+272018	-1259	-4634	+328376	+207634
-356	-4612	+41350	+816488	-1079	-3995	+40085	+765119	-380	-980	+6910	+132344	-192	-10608	+6738	+289374	-1214	-5196	+280626	+224268
-339	-5061	+32085	+851433	-1025	-4351	+30137	+789697	-363	-1077	+5499	+138687	-183	-11571	+5223	+300219	-1163	-5718	+228096	+236058
-320	-5465	+22129	+865084	-965	-4669	+19527	+794857	-344	-1165	+3979	+141772	-173	-12439	+3603	+304129	-1106	-6195	+171381	+242628
-299	-5819	+11615	+856405	-899	-4945	+8395	+779815	-323	-1243	+2369	+141427	-162	-13202	+1898	+300794	-1043	-6622	+111146	+243683
-276	-6118	+690	+824780	-827	-5175	-3105	+744165	-300	-1310	+690	+137540	-150	-13850	+130	+290030	-974	-6994	+48126	+239018
-251	-6357	-10485	+770055	-749	-5355	-14805	+687915	-275	-1365	-1035	+130065	-137	-14373	-1677	+271791	-899	-7306	-16874	+228527
-224	-6531	-21735	+692580	-665	-5481	-26523	+611523	-248	-1407	-2781	+119028	-123	-14761	-3497	+246181	-818	-7553	-82979	+212212
-195	-6635	-32871	+593251	-575	-5549	-38063	+515933	-219	-1435	-4521	+104533	-108	-15004	-5302	+213466	-731	-7730	-149244	+190192
-164	-6664	-43690	+473552	-479	-5555	-49215	+402611	-188	-1448	-6226	+86768	-92	-15092	-7062	+174086	-638	-7832	-214654	+162712
-131	-6613	-53975	+335597	-377	-5495	-59755	+273581	-155	-1445	-7865	+66011	-75	-15015	-8745	+128667	-539	-7854	-278124	+130152
-96	-6477	-63495	+182172	-269	-5365	-69445	+131461	-120	-1425	-9405	+42636	-57	-14763	-10317	+78033	-434	-7791	-338499	+93036
-59	-6251	-72005	+16777	-155	-5161	-78033	-20501	-83	-1387	-10811	+17119	-38	-14326	-11742	+23218	-323	-7638	-394554	+52041
-20	-5930	-79246	-156332	-35	-4879	-85253	-178391	-44	-1330	-12046	-9956	-18	-13694	-12982	-34522	-206	-7390	-444994	+8006
+21	-5509	-84945	-332101	+91	-4515	-90825	-337593	-3	-1253	-13071	-37891	+3	-12857	-13997	-93697	-83	-7042	-488454	-38059
+64	-4983	-88815	-504636	+223	-4065	-94455	-492753	+40	-1155	-13845	-65868	+25	-11805	-14745	-152571	+46	-6589	-523499	-84964
+109	-4347	-90555	-667161	+361	-3525	-95835	-637743	+85	-1035	-14325	-92943	+48	-10528	-15182	-209150	+181	-6026	-548624	-131330
+156	-3596	-89850	-811976	+505	-2891	-94643	-765625	+132	-892	-14466	-118040	+72	-9016	-15262	-261170	+322	-5348	-562254	-175580
+205	-2725	-86371	-930415	+655	-2159	-90543	-868615	+181	-725	-14221	-139945	+97	-7259	-14937	-306085	+469	-4550	-562744	-215930
+256	-1729	-79775	-1012804	+811	-1325	-83185	-938047	+232	-533	-13541	-157300	+123	-5247	-14157	-341055	+622	-3627	-548379	-250380
+309	-603	-69705	-1048419	+973	-385	-72205	-964337	+285	-315	-12375	-168597	+150	-2970	-12870	-362934	+781	-2574	-517374	-276705
+364	+658	-55790	-1025444	+1141	+665	-57225	-936947	+340	-70	-10670	-172172	+178	-418	-11022	-368258	+946	-1386	-467874	-292446
+421	+2059	-37645	-930929	+1315	+1829	-37853	-844349	+397	+203	-8371	-166199	+207	+2419	-8557	-353233	+1117	-58	-397954	-294901
+480	+3605	-14871	-750748	+1495	+3111	-13683	-673989	+456	+505	-5421	-148684	+237	+5551	-5417	-313723	+1294	+1415	-305619	-281116
+541	+5301	+12945	-469557	+1681	+4515	+15705	-412251	+517	+837	-1761	-117459	+268	+8988	-1542	-245238	+1477	+3038	-188804	-247876
+604	+7152	+46230	-70752	+1873	+6045	+50745	-44421	+580	+1200	+2670	-70176	+300	+12740	+3130	-142922	+1666	+4816	-45374	-191696
+669	+9163	+85425	+463573	+2071	+7705	+91885	+445349	+645	+1595	+7935	-4301	+333	+16817	+8663	-1541	+1861	+6754	+126876	-108812
+736	+11339	+130985	+1152668	+2275	+9499	+139587	+1074077	+712	+2023	+14099	+82892	+367	+21229	+15123	+184529	+2062	+8857	+330221	+4828
+805	+13685	+183379	+2017169	+2485	+11431	+194327	+1859987	+781	+2485	+21229	+194327	+402	+25986	+22578	+421314	+2269	+11130	+567006	+153573
								+852	+2982	+29394	+333132	+438	+31098	+31098	+715254	+2482	+13578	+839646	+342078
																+2701	+16206	+1150626	+575313

Table XXIV. Calculation of Integrals from equally spaced Ordinates

Each panel over which the integral is required is divided by calculated ordinates into a number of sub-panels, usually an even number, and often a power of 2. The formula may involve only internal and terminal ordinates, or also adjacent external ordinates. Ordinates equidistant from the centre have equal coefficients. The central coefficient is the last given.

Number of sub-panels.	No external ordinates. Coefficients.	Divisor.	One external ordinate. Coefficients.	Divisor.
2	1, 4	6	−1, 34, 114	180
4	7, 32, 12	90	−4, 171, 612, 332	1890
6	41, 216, 27, 272	840	−9, 482, 1908, 774, 2090	8400

Equally we may use the central ordinate and its even differences with the following coefficients:

Number of sub-panels.	δ^0	δ^2	δ^4	δ^6	δ^8	δ^{10}
2	1	0·1$\dot{6}$	−·00$\dot{5}$	·0006$\dot{6}$ 1375$\dot{}$	−·00010 141	·00001 76
4	1	0·6$\dot{6}$	·07$\dot{7}$	−·00211 6402$\dot{}$	·00022 928	−·00003 31
6	1	1·50	·550	·0488$\dot{0}$ 952$\dot{3}$	−·00107 143	·00010 28
8	1	2·6$\dot{6}$	1·91$\dot{1}$	·4867$\dot{7}$ 248$\dot{6}$	·03488 536	−·00063 28
10	1	4·1$\dot{6}$	4·86$\dot{1}$	2·2784$\dot{3}$ 915$\dot{3}$	·44587 743	·02683 41

When the number of sub-panels is large, Gregory's formula, in terms of terminal differences, is valuable:

$$\int_0^r f(x)dx-(\tfrac{1}{2}f_0+f_1+f_2\ \ldots\ +f_{r-1}+\tfrac{1}{2}f_r)$$

$$= -a_1(\Delta f_{r-1}-\Delta f_0)-a_2(\Delta^2 f_{r-2}+\Delta^2 f_0)-a_3(\Delta^3 f_{r-3}-\Delta^3 f_0)-a_4(\Delta^4 f_{r-4}+\Delta^4 f_0)\ \ldots$$

where a_r is the coefficient of x^r in the expansion of $-1/\log(1-x)$.

Values of a_1 to a_{16}

1	·0833$\dot{3}$	5	·01426 91799	9	·00678 585	13	·00421 5
2	·0416$\dot{6}$	6	·01136 73942	10	·00592 406	14	·00382 7
3	·0263$\dot{8}$	7	·00935 6537	11	·00523 67	15	·00350
4	·01875	8	·00789 2554	12	·00467 75	16	·00321

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
10	·00000	00432	00860	01284	01703	—	—	—	—	—	43	85	128	170	213
	—	—	—	—	—	02119	02531	02938	03342	03743	41	81	122	162	203
11	·04139	04532	04922	05308	05690	—	—	—	—	—	39	78	116	155	194
	—	—	—	—	—	06070	06446	06819	07188	07555	37	74	111	148	186
12	·07918	08279	08636	08991	09342	—	—	—	—	—	36	71	107	142	178
	—	—	—	—	—	09691	10037	10380	10721	11059	34	68	103	137	171
13	·11394	11727	12057	12385	12710	—	—	—	—	—	33	66	99	132	164
	—	—	—	—	—	13033	13354	13672	13988	14301	32	63	95	127	158
14	·14613	14922	15229	15534	15836	—	—	—	—	—	31	61	92	122	153
	—	—	—	—	—	16137	16435	16732	17026	17319	30	59	89	118	148
15	·17609	17898	18184	18469	18752	—	—	—	—	—	29	57	86	114	143
	—	—	—	—	—	19033	19312	19590	19866	20140	28	55	83	111	138
16	·20412	20683	20952	21219	21484	—	—	—	—	—	27	54	80	107	134
	—	—	—	—	—	21748	22011	22272	22531	22789	26	52	78	104	130
17	·23045	23300	23553	23805	24055	—	—	—	—	—	25	50	76	101	126
	—	—	—	—	—	24304	24551	24797	25042	25285	25	49	74	98	123
18	·25527	25768	26007	26245	26482	—	—	—	—	—	24	48	72	96	119
	—	—	—	—	—	26717	26951	27184	27416	27646	23	46	70	93	116
19	·27875	28103	28330	28556	28780	—	—	—	—	—	23	45	69	90	113
	—	—	—	—	—	29003	29226	29447	29667	29885	22	44	66	88	110
20	·30103	30320	30535	30750	30963	31175	31387	31597	31806	32015	21	42	64	85	106
21	·32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	40	61	81	101
22	·34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	19	39	58	77	97
23	·36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93
24	·38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18	36	53	71	89
25	·39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	85
26	·41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16	33	49	66	82
27	·43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79
28	·44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15	31	46	61	76
29	·46240	46389	46538	46687	46835	46982	47129	47276	47422	47567	15	29	44	59	74
30	·47712	47857	48001	48144	48287	48430	48572	48714	48855	48996	14	29	43	57	71
31	·49136	49276	49415	49554	49693	49831	49969	50106	50243	50379	14	28	41	55	69
32	·50515	50651	50786	50920	51055	51188	51322	51455	51587	51720	13	27	40	54	67
33	·51851	51983	52114	52244	52375	52504	52634	52763	52892	53020	13	26	39	52	65
34	·53148	53275	53403	53529	53656	53782	53908	54033	54158	54283	13	25	38	50	63
35	·54407	54531	54654	54777	54900	55023	55145	55267	55388	55509	12	24	37	49	61
36	·55630	55751	55871	55991	56110	56229	56348	56467	56585	56703	12	24	36	48	60
37	·56820	56937	57054	57171	57287	57403	57519	57634	57749	57864	12	23	35	46	58
38	·57978	58092	58206	58320	58433	58546	58659	58771	58883	58995	11	23	34	45	56
39	·59106	59218	59329	59439	59550	59660	59770	59879	59988	60097	11	22	33	44	55
40	·60206	60314	60423	60531	60638	60746	60853	60959	61066	61172	11	21	32	43	54
41	·61278	61384	61490	61595	61700	61805	61909	62014	62118	62221	10	21	31	42	52
42	·62325	62428	62531	62634	62737	62839	62941	63043	63144	63246	10	20	31	41	51
43	·63347	63448	63548	63649	63749	63849	63949	64048	64147	64246	10	20	30	40	50
44	·64345	64444	64542	64640	64738	64836	64933	65031	65128	65225	10	20	29	39	49
45	·65321	65418	65514	65610	65706	65801	65896	65992	66087	66181	10	19	29	38	48
46	·66276	66370	66464	66558	66652	66745	66839	66932	67025	67117	9	19	28	37	47
47	·67210	67302	67394	67486	67578	67669	67761	67852	67943	68034	9	18	27	37	46
48	·68124	68215	68305	68395	68485	68574	68664	68753	68842	68931	9	18	27	36	45
49	·69020	69108	69197	69285	69373	69461	69548	69636	69723	69810	9	18	26	35	44

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
50	·69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	9	17	26	34	43
51	·70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	8	17	25	34	42
52	·71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	8	17	25	33	41
53	·72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	8	16	24	32	41
54	·73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	8	16	24	32	40
55	·74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39
56	·74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	38
57	·75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	38
58	·76343	·76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37
59	·77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37
60	·77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36
61	·78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	35
62	·79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35
63	·79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	21	27	34
64	·80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7	13	20	27	34
65	·81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7	13	20	27	33
66	·81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7	13	20	26	33
67	·82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6	13	19	26	32
68	·83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6	13	19	25	32
69	·83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6	13	19	25	31
70	·84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6	12	19	25	31
71	·85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6	12	18	24	30
72	·85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6	12	18	24	30
73	·86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6	12	18	24	30
74	·86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6	12	17	23	29
75	·87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	6	12	17	23	29
76	·88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	6	11	17	23	28
77	·88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	6	11	17	22	28
78	·89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	6	11	17	22	28
79	·89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	5	11	16	22	27
80	·90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	5	11	16	22	27
81	·90849	90902	90956	91009	91062	91116	91169	91222	91275	91328	5	11	16	21	27
82	·91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	5	11	16	21	26
83	·91908	91960	92012	92065	92117	92169	92221	92273	92324	92376	5	10	16	21	26
84	·92428	92480	92531	92583	92634	92686	92737	92788	92840	92891	5	10	15	21	26
85	·92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	5	10	15	20	25
86	·93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	5	10	15	20	25
87	·93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	5	10	15	20	25
88	·94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	5	10	15	20	25
89	·94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	5	10	15	19	24
90	·95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	5	10	14	19	24
91	·95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	5	10	14	19	24
92	·96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	5	9	14	19	24
93	·96848	96895	96942	96988	97035	97081	97128	97174	97220	97267	5	9	14	19	23
94	·97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	5	9	14	18	23
95	·97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	5	9	14	18	23
96	·98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	4	9	14	18	22
97	·98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	4	9	13	18	22
98	·99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	4	9	13	18	22
99	·99564	99607	99651	99695	99739	99782	99826	99870	99913	99957	4	9	13	17	22

Table XXVI. Natural Logarithms

Negative Values, Decreasing

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
·10	2·30259	29263	28278	27303	26336	—	—	—	—	—	98	196	294	392	490
	—	—	—	—	—	25379	24432	23493	22562	21641	93	187	280	374	467
·11	2·20727	19823	18926	18037	17156	—	—	—	—	—	89	179	268	357	446
	—	—	—	—	—	16282	15417	14558	13707	12863	85	171	256	342	427
·12	2·12026	11196	10373	09557	08747	—	—	—	—	—	82	164	246	328	410
	—	—	—	—	—	07944	07147	06357	05573	04794	79	158	236	315	394
·13	2·04022	03256	02495	01741	00992	—	—	—	—	—	76	152	227	303	379
	—	—	—	—	—	00248	99510	98777	98050	97328	73	146	219	292	365
·14	1·96611	95900	95193	94491	93794	—	—	—	—	—	70	141	211	282	352
	—	—	—	—	—	93102	92415	91732	91054	90381	68	136	204	272	340
·15	1·89712	89048	88387	87732	87080	—	—	—	—	—	66	132	197	263	329
	—	—	—	—	—	86433	85790	85151	84516	83885	64	127	191	255	318
·16	1·83258	82635	82016	81401	80789	—	—	—	—	—	62	123	185	247	309
	—	—	—	—	—	80181	79577	78976	78379	77786	60	120	180	240	299
·17	1·77196	76609	76026	75446	74870	—	—	—	—	—	58	116	174	233	291
	—	—	—	—	—	74297	73727	73161	72597	72037	56	113	170	226	282
·18	1·71480	70926	70375	69827	69282	—	—	—	—	—	55	110	165	220	275
	—	—	—	—	—	68740	68201	67665	67131	66601	53	107	160	214	267
·19	1·66073	65548	65026	64507	63990	—	—	—	—	—	52	104	156	208	260
	—	—	—	—	—	63476	62964	62455	61949	61445	51	102	152	203	254
·20	1·60944	[illegible]	59949	59455	58964	58475	57988	57504	57022	56542	49	98	147	196	245
·21	1·56065	[illegible]	55117	54646	54178	53712	53248	52786	52326	51868	47	93	140	187	233
·22	1·51413	50959	50508	50058	49611	49165	48722	48281	47841	47403	45	89	134	178	223
·23	1·46968	46534	46102	45672	45243	44817	44392	43970	43548	43129	43	85	128	171	213
·24	1·42712	42296	41882	41469	41059	40650	40242	39837	39433	39030	41	82	123	164	205
·25	1·38629	38230	37833	37437	37042	36649	36258	35868	35480	35093	39	79	118	157	196
·26	1·34707	34323	33941	33560	33181	32803	32426	32051	31677	31304	38	76	113	151	189
·27	1·30933	30564	30195	29828	29463	29098	28735	28374	28013	27654	36	73	109	146	182
·28	1·27297	26940	26585	26231	25878	25527	25176	24827	24479	24133	35	70	105	141	176
·29	1·23787	23443	23100	22758	22418	22078	21740	21402	21066	20731	34	68	102	136	170
·30	1·20397	20065	19733	19402	19073	18744	18417	18091	17766	17441	33	66	99	131	164
·31	1·17118	16796	16475	16155	15836	15518	15201	14885	14570	14256	32	64	95	127	159
·32	1·13943	13631	13320	13010	12701	12393	12086	11780	11474	11170	31	62	92	123	154
·33	1·10866	10564	10262	09961	09661	09362	09064	08767	08471	08176	30	60	90	120	149
·34	1·07881	07587	07294	07002	06711	06421	06132	05843	05555	05268	29	58	87	116	145
·35	1·04982	04697	04412	04129	03846	03564	03282	03002	02722	02443	28	56	85	113	141
·36	1·02165	01888	01611	01335	01060	00786	00512	00239	99967	99696	27	55	82	110	137
·37	0·99425	99155	98886	98618	98350	98083	97817	97551	97286	97022	27	53	80	107	134
·38	0·96758	96496	96233	95972	95711	95451	95192	94933	94675	94418	26	52	78	104	130
·39	0·94161	93905	93649	93395	93140	92887	92634	92382	92130	91879	25	51	76	101	127
·40	0·91629	91379	91130	90882	90634	90387	90140	89894	89649	89404	25	49	74	99	124
·41	0·89160	88916	88673	88431	88189	87948	87707	87467	87227	86988	24	48	72	96	121
·42	0·86750	86512	86275	86038	85802	85567	85332	85097	84863	84630	24	47	71	94	118
·43	0·84397	84165	83933	83702	83471	83241	83011	82782	82554	82326	23	46	69	92	115
·44	0·82098	81871	81645	81419	81193	80968	80744	80520	80296	80073	22	45	68	90	112
·45	0·79851	79629	79407	79186	78966	78746	78526	78307	78089	77871	22	44	66	88	110
·46	0·77653	77436	77219	77003	76787	76572	76357	76143	75929	75715	22	43	65	86	108
·47	0·75502	75290	75078	74866	74655	74444	74234	74024	73814	73605	21	42	63	84	105
·48	0·73397	73189	72981	72774	72567	72361	72155	71949	71744	71539	21	41	62	83	103
·49	0·71335	71131	70928	70725	70522	70320	70118	69917	69716	69515	20	40	61	81	101

$\log_e 10 = 2{\cdot}30259$, $\log_e 100 = 4{\cdot}60517$, $\log_e 1000 = 6{\cdot}90776$

TABLE XXVI. NATURAL LOGARITHMS—*continued*

Negative Values, Decreasing

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
·50	0·69315	69115	68916	68717	68518	68320	68122	67924	67727	67531	20	40	59	79	99
·51	0·67334	67139	66943	66748	66553	66359	66165	65971	65778	65585	19	39	58	78	97
·52	0·65393	65201	65009	64817	64626	64436	64245	64055	63866	63677	19	38	57	76	95
·53	0·63488	63299	63111	62923	62736	62549	62362	62176	61990	61804	19	37	56	75	94
·54	0·61619	61434	61249	61065	60881	60697	60514	60331	60148	59966	18	37	55	73	92
·55	0·59784	59602	59421	59240	59059	58879	58699	58519	58340	58161	18	36	54	72	90
·56	0·57982	57803	57625	57448	57270	57093	56916	56740	56563	56387	18	35	53	71	89
·57	0·56212	56037	55862	55687	55513	55339	55165	54991	54818	54645	17	35	52	70	87
·58	0·54473	54300	54128	53957	53785	53614	53444	53273	53103	52933	17	34	51	68	86
·59	0·52763	52594	52425	52256	52088	51919	51751	51584	51416	51249	17	34	50	67	84
·60	0·51083	50916	50750	50584	50418	50253	50088	49923	49758	49594	17	33	50	66	83
·61	0·49430	49266	49102	48939	48776	48613	48451	48289	48127	47965	16	33	49	65	81
·62	0·47804	47642	47482	47321	47160	47006	46840	46681	46522	46362	16	32	48	64	80
·63	0·46204	46045	45887	45728	45571	45413	45256	45099	44942	44785	16	32	47	63	79
·64	0·44629	44473	44317	44161	44006	43850	43696	43541	43386	43232	16	31	47	62	78
·65	0·43078	42925	42771	42618	42465	42312	42159	42007	41855	41703	15	31	46	61	76
·66	0·41552	41400	41249	41098	40947	40797	40647	40497	40347	40197	15	30	45	60	75
·67	0·40048	39899	39750	39601	39453	39304	39156	39008	38861	38713	15	30	44	59	74
·68	0·38566	38419	38273	38126	37980	37834	37688	37542	37397	37251	15	29	44	58	73
·69	0·37106	36962	36817	36673	36528	36384	36241	36097	35954	35810	14	29	43	58	72
·70	0·35668	35525	35382	35240	35098	34956	34814	34672	34531	34390	14	[illegible]	43	57	71
·71	0·34249	34108	33968	33827	33687	33547	33408	33268	33129	32989	14	[illegible]	42	56	70
·72	0·32850	32712	32573	32435	32296	32158	32021	31883	31745	31608	14	28	41	55	69
·73	0·31471	31334	31197	31061	30925	30788	30653	30517	30381	30246	14	27	41	54	68
·74	0·30111	29975	29841	29706	29571	29437	29303	29169	29035	28902	13	27	40	54	67
·75	0·28768	28635	28502	28369	28236	28104	27971	27839	27707	27575	13	27	40	53	66
·76	0·27444	27312	27181	27050	26919	26788	26657	26527	26397	26266	13	26	39	52	65
·77	0·26136	26007	25877	25748	25618	25489	25360	25231	25103	24974	13	26	39	52	65
·78	0·24846	24718	24590	24462	24335	24207	24080	23953	23826	23699	13	25	38	51	64
·79	0·23572	23446	23319	23193	23067	22941	22816	22690	22565	22439	13	25	38	50	63
·80	0·22314	22189	22065	21940	21816	21691	21567	21443	21319	21196	12	25	37	50	62
·81	0·21072	20949	20825	20702	20579	20457	20334	20212	20089	19967	12	25	37	49	61
·82	0·19845	19723	19601	19480	19358	19237	19116	18995	18874	18754	12	24	36	49	61
·83	0·18633	18513	18392	18272	18152	18032	17913	17793	17674	17555	12	24	36	48	60
·84	0·17435	17316	17198	17079	16960	16842	16724	16605	16487	16370	12	24	36	47	59
·85	0·16252	16134	16017	15900	15782	15665	15548	15432	15315	15199	12	23	35	47	58
·86	0·15082	14966	14850	14734	14618	14503	14387	14272	14156	14041	12	23	35	46	58
·87	0·13926	13811	13697	13582	13467	13353	13239	13125	13011	12897	11	23	34	46	57
·88	0·12783	12670	12556	12443	12330	12217	12104	11991	11878	11766	11	23	34	45	56
·89	0·11653	11541	11429	11317	11205	11093	10981	10870	10759	10647	11	22	34	45	56
·90	0·10536	10425	10314	10203	10093	09982	09872	09761	09651	09541	11	22	33	44	55
·91	0·09431	09321	09212	09102	08992	08883	08774	08665	08556	08447	11	22	33	44	55
·92	0·08338	08230	08121	08013	07904	07796	07688	07580	07472	07365	11	22	32	43	54
·93	0·07257	07150	07042	06935	06828	06721	06614	06507	06401	06294	11	21	32	43	54
·94	0·06188	06081	05975	05869	05763	05657	05551	05446	05340	05235	11	21	32	42	53
·95	0·05129	05024	04919	04814	04709	04604	04500	04395	04291	04186	10	21	31	42	52
·96	0·04082	03978	03874	03770	03666	03563	03459	03356	03252	03149	10	21	31	41	52
·97	0·03046	02943	02840	02737	02634	02532	02429	02327	02225	02122	10	21	31	41	51
·98	0·02020	01918	01816	01715	01613	01511	01410	01309	01207	01106	10	20	30	41	51
·99	0·01005	00904	00803	00702	00602	00501	00401	00300	00200	00100	10	20	30	40	50

$\log_e 10{,}000 = 9{\cdot}21034$, $\log_e 100{,}000 = 11{\cdot}51293$, $\log_e 1{,}000{,}000 = 13{\cdot}81551$

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
1·0	0·00000	00995	01980	02956	03922	—	—	—	—	—	98	196	294	392	490
	—	—	—	—	—	04879	05827	06766	07696	08618	93	187	280	374	467
1·1	0·09531	10436	11333	12222	13103	—	—	—	—	—	89	179	268	357	446
	—	—	—	—	—	13976	14842	15700	16551	17395	85	171	256	342	427
1·2	0·18232	19062	19885	20701	21511	—	—	—	—	—	82	164	246	328	410
	—	—	—	—	—	22314	23111	23902	24686	25464	79	158	236	315	394
1·3	0·26236	27003	27763	28518	29267	—	—	—	—	—	76	152	227	303	379
	—	—	—	—	—	30010	30748	31481	32208	32930	73	146	219	292	365
1·4	0·33647	34359	35066	35767	36464	—	—	—	—	—	70	141	211	282	352
	—	—	—	—	—	37156	37844	38526	39204	39878	68	136	204	272	340
1·5	0·40547	41211	41871	42527	43178	—	—	—	—	—	66	132	197	263	329
	—	—	—	—	—	43825	44469	45108	45742	46373	64	127	191	255	318
1·6	0·47000	47623	48243	48858	49470	—	—	—	—	—	62	124	185	247	309
	—	—	—	—	—	50078	50682	51282	51879	52473	60	120	180	240	299
1·7	0·53063	53649	54232	54812	55389	—	—	—	—	—	58	116	174	233	291
	—	—	—	—	—	55962	56531	57098	57661	58222	56	113	169	226	282
1·8	0·58779	59333	59884	60432	60977	—	—	—	—	—	55	110	165	220	275
	—	—	—	—	—	61519	62058	62594	63127	63658	53	107	160	214	267
1·9	0·64185	64710	65233	65752	66269	—	—	—	—	—	52	104	156	208	260
	—	—	—	—	—	66783	67294	67803	68310	68813	51	102	152	203	254
2·0	0·69315	69813	70310	70804	71295	71784	72271	72755	73237	73716	49	98	147	196	244
2·1	0·74194	74669	75142	75612	76081	76547	77011	77473	77932	78390	47	93	140	186	233
2·2	0·78846	79299	79751	80200	80648	81093	81536	81978	82418	82855	45	89	134	178	223
2·3	0·83291	83725	84157	84587	85015	85442	85866	86289	86710	87129	43	85	128	171	213
2·4	0·87547	87963	88377	88789	89200	89609	90016	90422	90826	91228	41	82	123	164	204
2·5	0·91629	92028	92426	92822	93216	93609	94001	94391	94779	95166	39	79	118	157	196
2·6	0·95551	95935	96317	96698	97078	97456	97833	98208	98582	98954	38	76	113	151	189
2·7	0·99325	99695	**00063**	**00430**	**00796**	**01160**	**01523**	**01885**	**02245**	**02604**	36	73	109	146	182
2·8	1·02962	03318	03674	04028	04380	04732	05082	05431	05779	06126	35	70	105	141	176
2·9	1·06471	06815	07158	07500	07841	08181	08519	08856	09192	09527	34	68	102	136	170
3·0	1·09861	10194	10526	10856	11186	11514	11841	12168	12493	12817	33	66	99	131	164
3·1	1·13140	13462	13783	14103	14422	14740	15057	15373	15688	16002	32	64	95	127	159
3·2	1·16315	16627	16938	17248	17557	17865	18173	18479	18784	19089	31	62	92	123	154
3·3	1·19392	19695	19996	20297	20597	20896	21194	21491	21788	22083	30	60	90	120	150
3·4	1·22378	22671	22964	23256	23547	23837	24127	24415	24703	24990	29	58	87	116	145
3·5	1·25276	25562	25846	26130	26413	26695	26976	27257	27536	27815	28	56	85	113	141
3·6	1·28093	28371	28647	28923	29198	29473	29746	30019	30291	30563	27	55	82	110	137
3·7	1·30833	31103	31372	31641	31909	32176	32442	32708	32972	33237	27	53	80	107	134
3·8	1·33500	33763	34025	34286	34547	34807	35067	35325	35584	35841	26	52	78	104	130
3·9	1·36098	36354	36609	36864	37118	37372	37624	37877	38128	38379	25	51	76	101	127
4·0	1·38629	38879	39128	39377	39624	39872	40118	40364	40610	40854	25	49	74	99	124
4·1	1·41099	41342	41585	41828	42070	42311	42552	42792	43031	43270	24	48	72	96	121
4·2	1·43508	43746	43984	44220	44456	44692	44927	45161	45395	45629	24	47	71	94	118
4·3	1·45862	46094	46326	46557	46787	47018	47247	47476	47705	47933	23	46	69	92	115
4·4	1·48160	48387	48614	48840	49065	49290	49515	49739	49962	50185	22	45	68	90	112
4·5	1·50408	50630	50851	51072	51293	51513	51732	51951	52170	52388	22	44	66	88	110
4·6	1·52606	52823	53039	53256	53471	53687	53902	54116	54330	54543	22	43	65	86	108
4·7	1·54756	54969	55181	55393	55604	55814	56025	56235	56444	56653	21	42	63	84	105
4·8	1·56862	57070	57277	57485	57691	57898	58104	58309	58515	58719	21	41	62	83	103
4·9	1·58924	59127	59331	59534	59737	59939	60141	60342	60543	60744	20	40	61	81	101

$\log_e 10 = 2{\cdot}30259$, $\log_e 100 = 4{\cdot}60517$, $\log_e 1000 = 6{\cdot}90776$

TABLE XXVI. NATURAL LOGARITHMS—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
5·0	1·60944	61144	61343	61542	61741	61939	62137	62334	62531	62728	20	40	59	79	99
5·1	1·62924	63120	63315	63511	63705	63900	64094	64287	64481	64673	19	39	58	78	97
5·2	1·64866	65058	65250	65441	65632	65823	66013	66203	66393	66582	19	38	57	76	95
5·3	1·66771	66959	67147	67335	67523	67710	67896	68083	68269	68455	19	37	56	75	94
5·4	1·68640	68825	69010	69194	69378	69562	69745	69928	70111	70293	18	37	55	73	92
5·5	1·70475	70656	70838	71019	71199	71380	71560	71740	71919	72098	18	36	54	72	90
5·6	1·72277	72455	72633	72811	72988	73166	73342	73519	73695	73871	18	35	53	71	89
5·7	1·74047	74222	74397	74572	74746	74920	75094	75267	75440	75613	17	35	52	70	87
5·8	1·75786	75958	76130	76302	76473	76644	76815	76985	77156	77326	17	34	51	68	86
5·9	1·77495	77665	77834	78002	78171	78339	78507	78675	78842	79009	17	34	50	67	84
6·0	1·79176	79342	79509	79675	79840	80006	80171	80336	80500	80665	17	33	50	66	83
6·1	1·80829	80993	81156	81319	81482	81645	81808	81970	82132	82294	16	33	49	65	81
6·2	1·82455	82616	82777	82938	83098	83258	83418	83578	83737	83896	16	32	48	64	80
6·3	1·84055	84214	84372	84530	84688	84845	85003	85160	85317	85473	16	32	47	63	79
6·4	1·85630	85786	85942	86097	86253	86408	86563	86718	86872	87026	16	31	47	62	78
6·5	1·87180	87334	87487	87641	87794	87947	88099	88251	88403	88555	15	31	46	61	76
6·6	1·88707	88858	89010	89160	89311	89462	89612	89762	89912	90061	15	30	45	60	75
6·7	1·90211	90360	90509	90658	90806	90954	91102	91250	91398	91545	15	30	44	59	74
6·8	1·91692	91839	91986	92132	92279	92425	92571	92716	92862	93007	15	29	44	58	73
6·9	1·93152	93297	93442	93586	93730	93874	94018	94162	94305	94448	14	29	43	58	72
7·0	1·94591	94734	94876	95019	95161	95303	95445	95586	95727	95869	14	28	43	57	71
7·1	1·96009	96150	96291	96431	96571	96711	96851	96991	97130	97269	14	28	42	56	70
7·2	1·97408	97547	97685	97824	97962	98100	98238	98376	98513	98650	14	28	41	55	69
7·3	1·98787	98924	99061	99198	99334	99470	99606	99742	99877	**00013**	14	27	41	54	68
7·4	2·00148	00283	00418	00553	00687	00821	00956	01089	01223	01357	13	27	40	54	67
7·5	2·01490	01624	01757	01890	02022	02155	02287	02419	02551	02683	13	27	40	53	66
7·6	2·02815	02946	03078	03209	03340	03471	03601	03732	03862	03992	13	26	39	52	65
7·7	2·04122	04252	04381	04511	04640	04769	04898	05027	05156	05284	13	26	39	52	65
7·8	2·05412	05540	05668	05796	05924	06051	06179	06306	06433	06560	13	25	38	51	64
7·9	2·06686	06813	06939	07065	07191	07317	07443	07568	07694	07819	13	25	38	50	63
8·0	2·07944	08069	08194	08318	08443	08567	08691	08815	08939	09063	12	25	37	50	62
8·1	2·09186	09310	09433	09556	09679	09802	09924	10047	10169	10291	12	25	37	49	61
8·2	2·10413	10535	10657	10779	10900	11021	11142	11263	11384	11505	12	24	36	49	61
8·3	2·11626	11746	11866	11986	12106	12226	12346	12465	12585	12704	12	24	36	48	60
8·4	2·12823	12942	13061	13180	13298	13417	13535	13653	13771	13889	12	24	36	47	59
8·5	2·14007	14124	14242	14359	14476	14593	14710	14827	14943	15060	12	23	35	47	58
8·6	2·15176	15292	15409	15524	15640	15756	15871	15987	16102	16217	12	23	35	46	58
8·7	2·16332	16447	16562	16677	16791	16905	17020	17134	17248	17361	11	23	34	46	57
8·8	2·17475	17589	17702	17816	17929	18042	18155	18267	18380	18493	11	23	34	45	56
8·9	2·18605	18717	18830	18942	19054	19165	19277	19389	19500	19611	11	22	34	45	56
9·0	2·19722	19834	19944	20055	20166	20276	20387	20497	20607	20717	11	22	33	44	55
9·1	2·20827	20937	21047	21157	21266	21375	21485	21594	21703	21812	11	22	33	44	55
9·2	2·21920	22029	22138	22246	22354	22462	22570	22678	22786	22894	11	22	32	43	54
9·3	2·23001	23109	23216	23324	23431	23538	23645	23751	23858	23965	11	21	32	43	54
9·4	2·24071	24177	24284	24390	24496	24601	24707	24813	24918	25024	11	21	32	42	53
9·5	2·25129	25234	25339	25444	25549	25654	25759	25863	25968	26072	10	21	31	42	52
9·6	2·26176	26280	26384	26488	26592	26696	26799	26903	27006	27109	10	21	31	41	52
9·7	2·27213	27316	27419	27521	27624	27727	27829	27932	28034	28136	10	21	31	41	51
9·8	2·28238	28340	28442	28544	28646	28747	28849	28950	29051	29152	10	20	30	41	51
9·9	2·29253	29354	29455	29556	29657	29757	29858	29958	30058	30158	10	20	30	40	50

$\log_e 10{,}000 = 9{\cdot}21034$, $\log_e 100{,}000 = 11{\cdot}51293$, $\log_e 1{,}000{,}000 = 13{\cdot}81551$

Table XXVI. Natural Logarithms—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
10	2·30259	31254	32239	33214	34181	—	—	—	—	—	98	196	294	392	490
	—	—	—	—	—	35138	36085	37024	37955	38876	93	187	280	374	467
11	2·39790	40695	41591	42480	43361	—	—	—	—	—	89	179	268	357	446
	—	—	—	—	—	44235	45101	45959	46810	47654	85	171	256	342	427
12	2·48491	49321	50144	50960	51770	—	—	—	—	—	82	164	246	328	410
	—	—	—	—	—	52573	53370	54160	54945	55723	79	158	236	315	394
13	2·56495	57261	58022	58776	59525	—	—	—	—	—	76	152	227	303	379
	—	—	—	—	—	60269	61007	61740	62467	63189	73	146	219	292	365
14	2·63906	64617	65324	66026	66723	—	—	—	—	—	70	141	211	282	352
	—	—	—	—	—	67415	68102	68785	69463	70136	68	136	204	272	340
15	2·70805	71469	72130	72785	73437	—	—	—	—	—	66	132	197	263	329
	—	—	—	—	—	74084	74727	75366	76001	76632	64	127	191	255	318
16	2·77259	77882	78501	79117	79728	—	—	—	—	—	62	124	185	247	309
	—	—	—	—	—	80336	80940	81541	82138	82731	60	120	180	240	299
17	2·83321	83908	84491	85071	85647	—	—	—	—	—	58	116	174	233	291
	—	—	—	—	—	86220	86790	87356	87920	88480	56	113	169	226	282
18	2·89037	89591	90142	90690	91235	—	—	—	—	—	55	110	165	220	275
	—	—	—	—	—	91777	92316	92852	93386	93916	53	107	160	214	267
19	2·94444	94969	95491	96011	96527	—	—	—	—	—	52	104	156	208	260
	—	—	—	—	—	97041	97553	98062	98568	99072	51	102	152	203	254
20	2·99573	**00072**	**00568**	**01062**	**01553**	**02042**	**02529**	**03013**	**03495**	**03975**	49	98	147	196	244
21	3·04452	04927	05400	05871	06339	06805	07269	07731	08191	08649	47	93	140	186	233
22	3·09104	09558	10009	10459	10906	11352	11795	12237	12676	13114	45	89	134	178	223
23	3·13549	13983	14415	14845	15274	15700	16125	16548	16969	17388	43	85	128	171	213
24	3·17805	18221	18635	19048	19458	19867	20275	20680	21084	21487	41	82	123	164	204
25	3·21888	22287	22684	23080	23475	23868	24259	24649	25037	25424	39	79	118	157	196
26	3·25810	26194	26576	26957	27336	27714	28091	28466	28840	29213	38	76	113	151	189
27	3·29584	29953	30322	30689	31054	31419	31782	32143	32504	32863	36	73	109	146	182
28	3·33220	33577	33932	34286	34639	34990	35341	35690	36038	36384	35	70	105	141	176
29	3·36730	37074	37417	37759	38099	38439	38777	39115	39451	39786	34	68	102	136	170
30	3·40120	40453	40784	41115	41444	41773	42100	42426	42751	43076	33	66	99	131	164
31	3·43399	43721	44042	44362	44681	44999	45316	45632	45947	46261	32	64	95	127	159
32	3·46574	46886	47197	47507	47816	48124	48431	48738	49043	49347	31	62	92	123	154
33	3·49651	49953	50255	50556	50856	51155	51453	51750	52046	52342	30	60	90	120	150
34	3·52636	52930	53223	53515	53806	54096	54385	54674	54962	55249	29	58	87	116	145
35	3·55535	55820	56105	56388	56671	56953	57235	57515	57795	58074	28	56	85	113	141
36	3·58352	58629	58906	59182	59457	59731	60005	60278	60550	60821	27	55	82	110	137
37	3·61092	61362	61631	61899	62167	62434	62700	62966	63231	63495	27	53	80	107	134
38	3·63759	64021	64284	64545	64806	65066	65325	65584	65842	66099	26	52	78	104	130
39	3·66356	66612	66868	67122	67377	67630	67883	68135	68387	68638	25	51	76	101	127
40	3·68888	69138	69387	69635	69883	70130	70377	70623	70868	71113	25	49	74	99	124
41	3·71357	71601	71844	72086	72328	72569	72810	73050	73290	73529	24	48	72	96	121
42	3·73767	74005	74242	74479	74715	74950	75185	75420	75654	75887	24	47	71	94	118
43	3·76120	76352	76584	76815	77046	77276	77506	77735	77963	78191	23	46	69	92	115
44	3·78419	78646	78872	79098	79324	79549	79773	79997	80221	80444	22	45	68	90	112
45	3·80666	80888	81110	81331	81551	81771	81991	82210	82428	82647	22	44	66	88	110
46	3·82864	83081	83298	83514	83730	83945	84160	84374	84588	84802	22	43	65	86	108
47	3·85015	85227	85439	85651	85862	86073	86283	86493	86703	86912	21	42	63	84	105
48	3·87120	87328	87536	87743	87950	88156	88362	88568	88773	88978	21	41	62	83	103
49	3·89182	89386	89589	89792	89995	90197	90399	90600	90801	91002	20	40	61	81	101

$\log_e 10 = 2{\cdot}30259$, $\log_e 100 = 4{\cdot}60517$, $\log_e 1000 = 6{\cdot}90776$

TABLE XXVI. NATURAL LOGARITHMS—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
50	3·91202	91402	91601	91801	91999	92197	92395	92593	92790	92986	20	40	59	79	99
51	3·93183	93378	93574	93769	93964	94158	94352	94546	94739	94932	19	39	58	78	97
52	3·95124	95316	95508	95700	95891	96081	96272	96462	96651	96840	19	38	57	76	95
53	3·97029	97218	97406	97594	97781	97968	98155	98341	98527	98713	19	37	56	75	94
54	3·98898	99083	99268	99452	99636	99820	**00003**	**00186**	**00369**	**00551**	18	37	55	73	92
55	4·00733	00915	01096	01277	01458	01638	01818	01998	02177	02356	18	36	54	72	90
56	4·02535	02714	02892	03069	03247	03424	03601	03777	03954	04130	18	35	53	71	89
57	4·04305	04480	04655	04830	05004	05178	05352	05526	05699	05872	17	35	52	70	87
58	4·06044	06217	06389	06560	06732	06903	07073	07244	07414	07584	17	34	51	68	86
59	4·07754	07923	08092	08261	08429	08598	08766	08933	09101	09268	17	34	50	67	84
60	4·09434	09601	09767	09933	10099	10264	10429	10594	10759	10923	17	33	50	66	83
61	4·11087	11251	11415	11578	11741	11904	12066	12228	12390	12552	16	33	49	65	81
62	4·12713	12875	13036	13196	13357	13517	13677	13836	13995	14155	16	32	48	64	80
63	4·14313	14472	14630	14789	14946	15104	15261	15418	15575	15732	16	32	47	63	79
64	4·15888	16044	16200	16356	16511	16667	16821	16976	17131	17285	16	31	47	62	78
65	4·17439	17592	17746	17899	18052	18205	18358	18510	18662	18814	15	31	46	61	76
66	4·18965	19117	19268	19419	19570	19720	19870	20020	20170	20320	15	30	45	60	75
67	4·20469	20618	20767	20916	21065	21213	21361	21509	21656	21804	15	30	44	59	74
68	4·21951	22098	22244	22391	22537	22683	22829	22975	23120	23266	15	29	44	58	73
69	4·23411	23555	23700	23844	23989	24133	24276	24420	24563	24707	14	29	43	58	72
70	4·24850	24992	25135	25277	25419	25561	25703	25845	25986	26127	14	28	43	57	71
71	4·26268	26409	26549	26690	26830	26970	27110	27249	27388	27528	14	28	42	56	70
72	4·27667	27805	27944	28082	28221	28359	28496	28634	28772	28909	14	28	41	55	69
73	4·29046	29183	29320	29456	29592	29729	29865	30000	30136	30271	14	27	41	54	68
74	4·30407	30542	30676	30811	30946	31080	31214	31348	31482	31615	13	27	40	54	67
75	4·31749	31882	32015	32148	32281	32413	32546	32678	32810	32942	13	27	40	53	66
76	4·33073	33205	33336	33467	33598	33729	33860	33990	34120	34251	13	26	39	52	65
77	4·34381	34510	34640	34769	34899	35028	35157	35286	35414	35543	13	26	39	52	65
78	4·35671	35799	35927	36055	36182	36310	36437	36564	36691	36818	13	25	38	51	64
79	4·36945	37071	37198	37324	37450	37576	37701	37827	37952	38078	13	25	38	50	63
80	4·38203	38328	38452	38577	38701	38826	38950	39074	39198	39321	12	25	37	50	62
81	4·39445	39568	39692	39815	39938	40060	40183	40305	40428	40550	12	25	37	49	61
82	4·40672	40794	40916	41037	41159	41280	41401	41522	41643	41763	12	24	36	49	61
83	4·41884	42004	42125	42245	42365	42485	42604	42724	42843	42963	12	24	36	48	60
84	4·43082	43201	43319	43438	43557	43675	43793	43912	44030	44147	12	24	36	47	59
85	4·44265	44383	44500	44617	44735	44852	44969	45085	45202	45318	12	23	35	47	58
86	4·45435	45551	45667	45783	45899	46014	46130	46245	46361	46476	12	23	35	46	58
87	4·46591	46706	46820	46935	47050	47164	47278	47392	47506	47620	11	23	34	46	57
88	4·47734	47847	47961	48074	48187	48300	48413	48526	48639	48751	11	23	34	45	56
89	4·48864	48976	49088	49200	49312	49424	49536	49647	49758	49870	11	22	34	45	56
90	4·49981	50092	50203	50314	50424	50535	50645	50756	50866	50976	11	22	33	44	55
91	4·51086	51196	51305	51415	51525	51634	51743	51852	51961	52070	11	22	33	44	55
92	4·52179	52287	52396	52504	52613	52721	52829	52937	53045	53152	11	22	32	43	54
93	4·53260	53367	53475	53582	53689	53796	53903	54010	54116	54223	11	21	32	43	54
94	4·54329	54436	54542	54648	54754	54860	54966	55071	55177	55282	11	21	32	42	53
95	4·55388	55493	55598	55703	55808	55913	56017	56122	56226	56331	10	21	31	42	52
96	4·56435	56539	56643	56747	56851	56954	57058	57161	57265	57368	10	21	31	41	52
97	4·57471	57574	57677	57780	57883	57985	58088	58190	58292	58395	10	21	31	41	51
98	4·58497	58599	58701	58802	58904	59006	59107	59209	59310	59411	10	20	30	41	51
99	4·59512	59613	59714	59815	59915	60016	60116	60217	60317	60417	10	20	30	40	50

$\log_e 10{,}000 = 9{\cdot}21034$, $\log_e 100{,}000 = 11{\cdot}51293$. $\log_e 1{,}000{,}000 = 13{\cdot}81551$

Table XXVII. Squares

	0	1	2	3	4	5	6	7	8	9
0	0	1	4	9	16	25	36	49	64	81
1	100	121	144	169	196	225	256	289	324	361
2	400	441	484	529	576	625	676	729	784	841
3	900	961	1024	1089	1156	1225	1296	1369	1444	1521
4	1600	1681	1764	1849	1936	2025	2116	2209	2304	2401
5	2500	2601	2704	2809	2916	3025	3136	3249	3364	3481
6	3600	3721	3844	3969	4096	4225	4356	4489	4624	4761
7	4900	5041	5184	5329	5476	5625	5776	5929	6084	6241
8	6400	6561	6724	6889	7056	7225	7396	7569	7744	7921
9	8100	8281	8464	8649	8836	9025	9216	9409	9604	9801
10	10000	10201	10404	10609	10816	11025	11236	11449	11664	11881
11	12100	12321	12544	12769	12996	13225	13456	13689	13924	14161
12	14400	14641	14884	15129	15376	15625	15876	16129	16384	16641
13	16900	17161	17424	17689	17956	18225	18496	18769	19044	19321
14	19600	19881	20164	20449	20736	21025	21316	21609	21904	22201
15	22500	22801	23104	23409	23716	24025	24336	24649	24964	25281
16	25600	25921	26244	26569	26896	27225	27556	27889	28224	28561
17	28900	29241	29584	29929	30276	30625	30976	31329	31684	32041
18	32400	32761	33124	33489	33856	34225	34596	34969	35344	35721
19	36100	36481	36864	37249	37636	38025	38416	38809	39204	39601
20	40000	40401	40804	41209	41616	42025	42436	42849	43264	43681
21	44100	44521	44944	45369	45796	46225	46656	47089	47524	47961
22	48400	48841	49284	49729	50176	50625	51076	51529	51984	52441
23	52900	53361	53824	54289	54756	55225	55696	56169	56644	57121
24	57600	58081	58564	59049	59536	60025	60516	61009	61504	62001
25	62500	63001	63504	64009	64516	65025	65536	66049	66564	67081
26	67600	68121	68644	69169	69696	70225	70756	71289	71824	72361
27	72900	73441	73984	74529	75076	75625	76176	76729	77284	77841
28	78400	78961	79524	80089	80656	81225	81796	82369	82944	83521
29	84100	84681	85264	85849	86436	87025	87616	88209	88804	89401
30	90000	90601	91204	91809	92416	93025	93636	94249	94864	95481
31	96100	96721	97344	97969	98596	99225	99856	100489	101124	101761
32	102400	103041	103684	104329	104976	105625	106276	106929	107584	108241
33	108900	109561	110224	110889	111556	112225	112896	113569	114244	114921
34	115600	116281	116964	117649	118336	119025	119716	120409	121104	121801
35	122500	123201	123904	124609	125316	126025	126736	127449	128164	128881
36	129600	130321	131044	131769	132496	133225	133956	134689	135424	136161
37	136900	137641	138384	139129	139876	140625	141376	142129	142884	143641
38	144400	145161	145924	146689	147456	148225	148996	149769	150544	151321
39	152100	152881	153664	154449	155236	156025	156816	157609	158404	159201
40	160000	160801	161604	162409	163216	164025	164836	165649	166464	167281
41	168100	168921	169744	170569	171396	172225	173056	173889	174724	175561
42	176400	177241	178084	178929	179776	180625	181476	182329	183184	184041
43	184900	185761	186624	187489	188356	189225	190096	190969	191844	192721
44	193600	194481	195364	196249	197136	198025	198916	199809	200704	201601
45	202500	203401	204304	205209	206116	207025	207936	208849	209764	210681
46	211600	212521	213444	214369	215296	216225	217156	218089	219024	219961
47	220900	221841	222784	223729	224676	225625	226576	227529	228484	229441
48	230400	231361	232324	233289	234256	235225	236196	237169	238144	239121
49	240100	241081	242064	243049	244036	245025	246016	247009	248004	249001

Exact squares of 4 figure numbers can be quickly calculated from the identity $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

TABLE XXVII. SQUARES—*continued*

	0	1	2	3	4	5	6	7	8	9
50	250000	251001	252004	253009	254016	255025	256036	257049	258064	259081
51	260100	261121	262144	263169	264196	265225	266256	267289	268324	269361
52	270400	271441	272484	273529	274576	275625	276676	277729	278784	279841
53	280900	281961	283024	284089	285156	286225	287296	288369	289444	290521
54	291600	292681	293764	294849	295936	297025	298116	299209	300304	301401
55	302500	303601	304704	305809	306916	308025	309136	310249	311364	312481
56	313600	314721	315844	316969	318096	319225	320356	321489	322624	323761
57	324900	326041	327184	328329	329476	330625	331776	332929	334084	335241
58	336400	337561	338724	339889	341056	342225	343396	344569	345744	346921
59	348100	349281	350464	351649	352836	354025	355216	356409	357604	358801
60	360000	361201	362404	363609	364816	366025	367236	368449	369664	370881
61	372100	373321	374544	375769	376996	378225	379456	380689	381924	383161
62	384400	385641	386884	388129	389376	390625	391876	393129	394384	395641
63	396900	398161	399424	400689	401956	403225	404496	405769	407044	408321
64	409600	410881	412164	413449	414736	416025	417316	418609	419904	421201
65	422500	423801	425104	426409	427716	429025	430336	431649	432964	434281
66	435600	436921	438244	439569	440896	442225	443556	444889	446224	447561
67	448900	450241	451584	452929	454276	455625	456976	458329	459684	461041
68	462400	463761	465124	466489	467856	469225	470596	471969	473344	474721
69	476100	477481	478864	480249	481636	483025	484416	485809	487204	488601
70	490000	491401	492804	494209	495616	497025	498436	499849	501264	502681
71	504100	505521	506944	508369	509796	511225	512656	514089	515524	516961
72	518400	519841	521284	522729	524176	525625	527076	528529	529984	531441
73	532900	534361	535824	537289	538756	540225	541696	543169	544644	546121
74	547600	549081	550564	552049	553536	555025	556516	558009	559504	561001
75	562500	564001	565504	567009	568516	570025	571536	573049	574564	576081
76	577600	579121	580644	582169	583696	585225	586756	588289	589824	591361
77	592900	594441	595984	597529	599076	600625	602176	603729	605284	606841
78	608400	609961	611524	613089	614656	616225	617796	619369	620944	622521
79	624100	625681	627264	628849	630436	632025	633616	635209	636804	638401
80	640000	641601	643204	644809	646416	648025	649636	651249	652864	654481
81	656100	657721	659344	660969	662596	664225	665856	667489	669124	670761
82	672400	674041	675684	677329	678976	680625	682276	683929	685584	687241
83	688900	690561	692224	693889	695556	697225	698896	700569	702244	703921
84	705600	707281	708964	710649	712336	714025	715716	717409	719104	720801
85	722500	724201	725904	727609	729316	731025	732736	734449	736164	737881
86	739600	741321	743044	744769	746496	748225	749956	751689	753424	755161
87	756900	758641	760384	762129	763876	765625	767376	769129	770884	772641
88	774400	776161	777924	779689	781456	783225	784996	786769	788544	790321
89	792100	793881	795664	797449	799236	801025	802816	804609	806404	808201
90	810000	811801	813604	815409	817216	819025	820836	822649	824464	826281
91	828100	829921	831744	833569	835396	837225	839056	840889	842724	844561
92	846400	848241	850084	851929	853776	855625	857476	859329	861184	863041
93	864900	866761	868624	870489	872356	874225	876096	877969	879844	881721
94	883600	885481	887364	889249	891136	893025	894916	896809	898704	900601
95	902500	904401	906304	908209	910116	912025	913936	915849	917764	919681
96	921600	923521	925444	927369	929296	931225	933156	935089	937024	938961
97	940900	942841	944784	946729	948676	950625	952576	954529	956484	958441
98	960400	962361	964324	966289	968256	970225	972196	974169	976144	978121
99	980100	982081	984064	986049	988036	990025	992016	994009	996004	998001

Thus $693{\cdot}3^2 = 480249 + 415{\cdot}8 + 0{\cdot}09 = 480664{\cdot}89$.

TABLE XXVIII. SQUARE ROOTS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
10	10000	10050	10100	10149	10198	10247	10296	10344	10392	10440	5	10	15	20	24
	31623	31780	31937	32094	32249	32404	32558	32711	32863	33015	15	31	46	62	77
11	10488	10536	10583	10630	10677	10724	10770	10817	10863	10909	5	9	14	19	23
	33166	33317	33466	33615	33764	33912	34059	34205	34351	34496	15	30	44	59	74
12	10954	11000	11045	11091	11136	11180	11225	11269	11314	11358	4	9	13	18	22
	34641	34785	34928	35071	35214	35355	35496	35637	35777	35917	14	28	43	57	71
13	11402	11446	11489	11533	11576	11619	11662	11705	11747	11790	4	9	13	17	22
	36056	36194	36332	36469	36606	36742	36878	37014	37148	37283	14	27	41	55	68
14	11832	11874	11916	11958	12000	12042	12083	12124	12166	12207	4	8	13	17	21
	37417	37550	37683	37815	37947	38079	38210	38341	38471	38601	13	26	39	53	66
15	12247	12288	12329	12369	12410	12450	12490	12530	12570	12610	4	8	12	16	20
	38730	38859	38987	39115	39243	39370	39497	39623	39749	39875	13	25	38	51	64
16	12649	12689	12728	12767	12806	12845	12884	12923	12961	13000	4	8	12	16	20
	40000	40125	40249	40373	40497	40620	40743	40866	40988	41110	12	25	37	49	62
17	13038	13077	13115	13153	13191	13229	13266	13304	13342	13379	4	8	11	15	19
	41231	41352	41473	41593	41713	41833	41952	42071	42190	42308	12	24	36	48	60
18	13416	13454	13491	13528	13565	13601	13638	13675	13711	13748	4	7	11	15	18
	42426	42544	42661	42778	42895	43012	43128	43243	43359	43474	12	23	35	47	58
19	13784	13820	13856	13892	13928	13964	14000	14036	14071	14107	4	7	11	14	18
	43589	43704	43818	43932	44045	44159	44272	44385	44497	44609	11	23	34	45	57
20	14142	14177	14213	14248	14283	14318	14353	14387	14422	14457	4	7	10	14	18
	44721	44833	44944	45056	45166	45277	45387	45497	45607	45717	11	22	33	44	55
21	14491	14526	14560	14595	14629	14663	14697	14731	14765	14799	3	7	10	14	17
	45826	45935	46043	46152	46260	46368	46476	46583	46690	46797	11	22	32	43	54
22	14832	14866	14900	14933	14967	15000	15033	15067	15100	15133	3	7	10	13	17
	46904	47011	47117	47223	47329	47434	47539	47645	47749	47854	11	21	32	42	53
23	15166	15199	15232	15264	15297	15330	15362	15395	15427	15460	3	7	10	13	16
	47958	48062	48166	48270	48374	48477	48580	48683	48785	48888	10	21	31	41	52
24	15492	15524	15556	15588	15620	15652	15684	15716	15748	15780	3	6	10	13	16
	48990	49092	49193	49295	49396	49497	49598	49699	49800	49900	10	20	30	40	51
25	15811	15843	15875	15906	15937	15969	16000	16031	16062	16093	3	6	9	13	16
	50000	50100	50200	50299	50398	50498	50596	50695	50794	50892	10	20	30	40	50
26	16125	16155	16186	16217	16248	16279	16310	16340	16371	16401	3	6	9	12	15
	50990	51088	51186	51284	51381	51478	51575	51672	51769	51865	10	19	29	39	49
27	16432	16462	16492	16523	16553	16583	16613	16643	16673	16703	3	6	9	12	15
	51962	52058	52154	52249	52345	52440	52536	52631	52726	52820	10	19	29	38	48
28	16733	16763	16793	16823	16852	16882	16912	16941	16971	17000	3	6	9	12	15
	52915	53009	53104	53198	53292	53385	53479	53572	53666	53759	9	19	28	38	47
29	17029	17059	17088	17117	17146	17176	17205	17234	17263	17292	3	6	9	12	15
	53852	53944	54037	54129	54222	54314	54406	54498	54589	54681	9	18	28	37	46

TABLE XXVIII. SQUARE ROOTS—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
30	17321	17349	17378	17407	17436	17464	17493	17521	17550	17578	3	6	9	11	14
	54772	54863	54955	55045	55136	55227	55317	55408	55498	55588	9	18	27	36	45
31	17607	17635	17664	17692	17720	17748	17776	17804	17833	17861	3	6	8	11	14
	55678	55767	55857	55946	56036	56125	56214	56303	56391	56480	9	18	27	36	45
32	17889	17916	17944	17972	18000	18028	18055	18083	18111	18138	3	6	8	11	14
	56569	56657	56745	56833	56921	57009	57096	57184	57271	57359	9	18	26	35	44
33	18166	18193	18221	18248	18276	18303	18330	18358	18385	18412	3	5	8	11	14
	57446	57533	57619	57706	57793	57879	57966	58052	58138	58224	9	17	26	35	43
34	18439	18466	18493	18520	18547	18574	18601	18628	18655	18682	3	5	8	11	14
	58310	58395	58481	58566	58652	58737	58822	58907	58992	59076	9	17	26	34	43
35	18708	18735	18762	18788	18815	18841	18868	18894	18921	18947	3	5	8	11	13
	59161	59245	59330	59414	59498	59582	59666	59749	59833	59917	8	17	25	34	42
36	18974	19000	19026	19053	19079	19105	19131	19157	19183	19209	3	5	8	10	13
	60000	60083	60166	60249	60332	60415	60498	60581	60663	60745	8	17	25	33	41
37	19235	19261	19287	19313	19339	19365	19391	19416	19442	19468	3	5	8	10	13
	60828	60910	60992	61074	61156	61237	61319	61400	61482	61563	8	16	25	33	41
38	19494	19519	19545	19570	19596	19621	19647	19672	19698	19723	3	5	8	10	13
	61644	61725	61806	61887	61968	62048	62129	62209	62290	62370	8	16	24	32	40
39	19748	19774	19799	19824	19849	19875	19900	19925	19950	19975	3	5	8	10	13
	62450	62530	62610	62690	62769	62849	62929	63008	63087	63166	8	16	24	32	40
40	20000	20025	20050	20075	20100	20125	20149	20174	20199	20224	2	5	7	10	12
	63246	63325	63403	63482	63561	63640	63718	63797	63875	63953	8	16	24	31	39
41	20248	20273	20298	20322	20347	20372	20396	20421	20445	20469	2	5	7	10	12
	64031	64109	64187	64265	64343	64420	64498	64576	64653	64730	8	16	23	31	39
42	20494	20518	20543	20567	20591	20616	20640	20664	20688	20712	2	5	7	10	12
	64807	64885	64962	65038	65115	65192	65269	65345	65422	65498	8	15	23	31	38
43	20736	20761	20785	20809	20833	20857	20881	20905	20928	20952	2	5	7	10	12
	65574	65651	65727	65803	65879	65955	66030	66106	66182	66257	8	15	23	30	38
44	20976	21000	21024	21048	21071	21095	21119	21142	21166	21190	2	5	7	10	12
	66332	66408	66483	66558	66633	66708	66783	66858	66933	67007	8	15	22	30	38
45	21213	21237	21260	21284	21307	21331	21354	21378	21401	21424	2	5	7	9	12
	67082	67157	67231	67305	67380	67454	67528	67602	67676	67750	7	15	22	30	37
46	21448	21471	21494	21517	21541	21564	21587	21610	21633	21656	2	5	7	9	12
	67823	67897	67971	68044	68118	68191	68264	68337	68411	68484	7	15	22	29	37
47	21679	21703	21726	21749	21772	21794	21817	21840	21863	21886	2	5	7	9	12
	68557	68629	68702	68775	68848	68920	68993	69065	69138	69210	7	15	22	29	36
48	21909	21932	21954	21977	22000	22023	22045	22068	22091	22113	2	5	7	9	11
	69282	69354	69426	69498	69570	69642	69714	69785	69857	69929	7	14	22	29	36
49	22136	22159	22181	22204	22226	22249	22271	22293	22316	22338	2	4	7	9	11
	70000	70071	70143	70214	70285	70356	70427	70498	70569	70640	7	14	21	28	36

TABLE XXVIII. SQUARE ROOTS—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
50	22361	22383	22405	22428	22450	22472	22494	22517	22539	22561	2	4	7	9	11
	70711	70781	70852	70922	70993	71063	71134	71204	71274	71344	7	14	21	28	35
51	22583	22605	22627	22650	22672	22694	22716	22738	22760	22782	2	4	7	9	11
	71414	71484	71554	71624	71694	71764	71833	71903	71972	72042	7	14	21	28	35
52	22804	22825	22847	22869	22891	22913	22935	22956	22978	23000	2	4	7	9	11
	72111	72180	72250	72319	72388	72457	72526	72595	72664	72732	7	14	21	28	34
53	23022	23043	23065	23087	23108	23130	23152	23173	23195	23216	2	4	6	9	11
	72801	72870	72938	73007	73075	73144	73212	73280	73348	73417	7	14	21	27	34
54	23238	23259	23281	23302	23324	23345	23367	23388	23409	23431	2	4	6	9	11
	73485	73553	73621	73689	73756	73824	73892	73959	74027	74095	7	14	20	27	34
55	23452	23473	23495	23516	23537	23558	23580	23601	23622	23643	2	4	6	8	11
	74162	74229	74297	74364	74431	74498	74565	74632	74699	74766	7	13	20	27	34
56	23664	23685	23707	23728	23749	23770	23791	23812	23833	23854	2	4	6	8	11
	74833	74900	74967	75033	75100	75166	75233	75299	75366	75432	7	13	20	27	33
57	23875	23896	23917	23937	23958	23979	24000	24021	24042	24062	2	4	6	8	10
	75498	75565	75631	75697	75763	75829	75895	75961	76026	76092	7	13	20	26	33
58	24083	24104	24125	24145	24166	24187	24207	24228	24249	24269	2	4	6	8	10
	76158	76223	76289	76354	76420	76485	76551	76616	76681	76746	7	13	20	26	33
59	24290	24310	24331	24352	24372	24393	24413	24434	24454	24474	2	4	6	8	10
	76811	76877	76942	77006	77071	77136	77201	77266	77330	77395	6	13	19	26	32
60	24495	24515	24536	24556	24576	24597	24617	24637	24658	24678	2	4	6	8	10
	77460	77524	77589	77653	77717	77782	77846	77910	77974	78038	6	13	19	26	32
61	24698	24718	24739	24759	24779	24799	24819	24839	24860	24880	2	4	6	8	10
	78102	78166	78230	78294	78358	78422	78486	78549	78613	78677	6	13	19	26	32
62	24900	24920	24940	24960	24980	25000	25020	25040	25060	25080	2	4	6	8	10
	78740	78804	78867	78930	78994	79057	79120	79183	79246	79310	6	13	19	25	32
63	25100	25120	25140	25159	25179	25199	25219	25239	25259	25278	2	4	6	8	10
	79373	79436	79498	79561	79624	79687	79750	79812	79875	79937	6	13	19	25	31
64	25298	25318	25338	25357	25377	25397	25417	25436	25456	25475	2	4	6	8	10
	80000	80062	80125	80187	80250	80312	80374	80436	80498	80561	6	12	19	25	31
65	25495	25515	25534	25554	25573	25593	25612	25632	25652	25671	2	4	6	8	10
	80623	80685	80747	80808	80870	80932	80994	81056	81117	81179	6	12	19	25	31
66	25690	25710	25729	25749	25768	25788	25807	25826	25846	25865	2	4	6	8	10
	81240	81302	81363	81425	81486	81548	81609	81670	81731	81792	6	12	18	25	31
67	25884	25904	25923	25942	25962	25981	26000	26019	26038	26058	2	4	6	8	10
	81854	81915	81976	82037	82098	82158	82219	82280	82341	82401	6	12	18	24	30
68	26077	26096	26115	26134	26153	26173	26192	26211	26230	26249	2	4	6	8	10
	82462	82523	82583	82644	82704	82765	82825	82885	82946	83006	6	12	18	24	30
69	26268	26287	26306	26325	26344	26363	26382	26401	26420	26439	2	4	6	8	10
	83066	83126	83187	83247	83307	83367	83427	83487	83546	83606	6	12	18	24	30
70	26458	26476	26495	26514	26533	26552	26571	26589	26608	26627	2	4	6	8	9
	83666	83726	83785	83845	83905	83964	84024	84083	84143	84202	6	12	18	24	30
71	26646	26665	26683	26702	26721	26739	26758	26777	26796	26814	2	4	6	7	9
	84261	84321	84380	84439	84499	84558	84617	84676	84735	84794	6	12	18	24	30
72	26833	26851	26870	26889	26907	26926	26944	26963	26981	27000	2	4	6	7	9
	84853	84912	84971	85029	85088	85147	85206	85264	85323	85381	6	12	18	23	29
73	27019	27037	27055	27074	27092	27111	27129	27148	27166	27185	2	4	6	7	9
	85440	85499	85557	85615	85674	85732	85790	85849	85907	85965	6	12	17	23	29
74	27203	27221	27240	27258	27276	27295	27313	27331	27350	27368	2	4	5	7	9
	86023	86081	86139	86197	86255	86313	86371	86429	86487	86545	6	12	17	23	29

TABLE XXVIII. SQUARE ROOTS—*continued*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5
75	27386	27404	27423	27441	27459	27477	27495	27514	27532	27550	2	4	5	7	9
	86603	86660	86718	86776	86833	86891	86948	87006	87063	87121	6	12	17	23	29
76	27568	27586	27604	27622	27641	27659	27677	27695	27713	27731	2	4	5	7	9
	87178	87235	87293	87350	87407	87464	87521	87579	87636	87693	6	11	17	23	29
77	27749	27767	27785	27803	27821	27839	27857	27875	27893	27911	2	4	5	7	9
	87750	87807	87864	87920	87977	88034	88091	88148	88204	88261	6	11	17	23	28
78	27928	27946	27964	27982	28000	28018	28036	28054	28071	28089	2	4	5	7	9
	88318	88374	88431	88487	88544	88600	88657	88713	88769	88826	6	11	17	23	28
79	28107	28125	28142	28160	28178	28196	28213	28231	28249	28267	2	4	5	7	9
	88882	88938	88994	89051	89107	89163	89219	89275	89331	89387	6	11	17	22	28
80	28284	28302	28320	28337	28355	28373	28390	28408	28425	28443	2	4	5	7	9
	89443	89499	89554	89610	89666	89722	89778	89833	89889	89944	6	11	17	22	28
81	28460	28478	28496	28513	28531	28548	28566	28583	28601	28618	2	4	5	7	9
	90000	90056	90111	90167	90222	90277	90333	90388	90443	90499	6	11	17	22	28
82	28636	28653	28671	28688	28705	28723	28740	28758	28775	28792	2	3	5	7	9
	90554	90609	90664	90719	90774	90830	90885	90940	90995	91049	6	11	16	22	28
83	28810	28827	28844	28862	28879	28896	28914	28931	28948	28965	2	3	5	7	9
	91104	91159	91214	91269	91324	91378	91433	91488	91542	91597	5	11	16	22	27
84	28983	29000	29017	29034	29052	29069	29086	29103	29120	29138	2	3	5	7	9
	91652	91706	91761	91815	91869	91924	91978	92033	92087	92141	5	11	16	22	27
85	29155	29172	29189	29206	29223	29240	29257	29275	29292	29309	2	3	5	7	9
	92195	92250	92304	92358	92412	92466	92520	92574	92628	92682	5	11	16	22	27
86	29326	29343	29360	29377	29394	29411	29428	29445	29462	29479	2	3	5	7	8
	92736	92790	92844	92898	92952	93005	93059	93113	93167	93220	5	11	16	22	27
87	29496	29513	29530	29547	29563	29580	29597	29614	29631	29648	2	3	5	7	8
	93274	93327	93381	93434	93488	93541	93595	93648	93702	93755	5	11	16	21	27
88	29665	29682	29698	29715	29732	29749	29766	29783	29799	29816	2	3	5	7	8
	93808	93862	93915	93968	94021	94074	94128	94181	94234	94287	5	11	16	21	27
89	29833	29850	29866	29883	29900	29917	29933	29950	29967	29983	2	3	5	7	8
	94340	94393	94446	94499	94552	94604	94657	94710	94763	94816	5	11	16	21	26
90	30000	30017	30033	30050	30067	30083	30100	30116	30133	30150	2	3	5	7	8
	94868	94921	94974	95026	95079	95131	95184	95237	95289	95341	5	11	16	21	26
91	30166	30183	30199	30216	30232	30249	30265	30282	30299	30315	2	3	5	7	8
	95394	95446	95499	95551	95603	95656	95708	95760	95812	95864	5	10	16	21	26
92	30332	30348	30364	30381	30397	30414	30430	30447	30463	30480	2	3	5	7	8
	95917	95969	96021	96073	96125	96177	96229	96281	96333	96385	5	10	16	21	26
93	30496	30512	30529	30545	30561	30578	30594	30610	30627	30643	2	3	5	7	8
	96437	96488	96540	96592	96644	96695	96747	96799	96850	96902	5	10	16	21	26
94	30659	30676	30692	30708	30725	30741	30757	30773	30790	30806	2	3	5	7	8
	96954	97005	97057	97108	97160	97211	97263	97314	97365	97417	5	10	15	21	26
95	30822	30838	30854	30871	30887	30903	30919	30935	30952	30968	2	3	5	6	8
	97468	97519	97570	97622	97673	97724	97775	97826	97877	97929	5	10	15	20	26
96	30984	31000	31016	31032	31048	31064	31081	31097	31113	31129	2	3	5	6	8
	97980	98031	98082	98133	98184	98234	98285	98336	98387	98438	5	10	15	20	25
97	31145	31161	31177	31193	31209	31225	31241	31257	31273	31289	2	3	5	6	8
	98489	98539	98590	98641	98691	98742	98793	98843	98894	98944	5	10	15	20	25
98	31305	31321	31337	31353	31369	31385	31401	31417	31432	31448	2	3	5	6	8
	98995	99045	99096	99146	99197	99247	99298	99348	99398	99448	5	10	15	20	25
99	31464	31480	31496	31512	31528	31544	31559	31575	31591	31607	2	3	5	6	8
	99499	99549	99599	99649	99700	99750	99800	99850	99900	99950	5	10	15	20	25

Table XXIX. Reciprocals

	0	1	2	3	4	5	6	7	8	9	Diffs.*
1·0	1·000000	990099	980392	970874	961538	952381	943396	934579	925926	917431	
	−9901	−9707	−9518	−9336	−9157	−8985	−8817	−8653	−8495	−8340	
1·1	·909091	900901	892857	884956	877193	869565	862069	854701	847458	840336	
	−8190	−8044	−7901	−7763	−7628	−7496	−7368	−7243	−7122	−7003	
1·2	·833333	826446	819672	813008	806452	800000	793651	787402	781250	775194	
	−6887	−6774	−6664	−6556	−6452	−6349	−6249	−6152	−6056	−5963	
1·3	·769231	763359	757576	751880	746269	740741	735294	729927	724638	719424	
	−5872	−5783	−5696	−5611	−5528	−5447	−5367	−5289	−5214	−5138	
1·4	·714286	709220	704225	699301	694444	689655	684932	680272	675676	671141	
	−5066	−4995	−4924	−4857	−4789	−4723	−4660	−4596	−4535	−4474	
1·5	·666667	662252	657895	653595	649351	645161	641026	636943	632911	628931	
	−4415	−4357	−4300	−4244	−4190	−4135	−4083	−4032	−3980	−3931	
1·6	·625000	621118	617284	613497	609756	606061	602410	598802	595238	591716	
	−3882	−3834	−3787	−3741	−3695	−3651	−3608	−3564	−3522	−3481	
1·7	·588235	584795	581395	578035	574713	571429	568182	564972	561798	558659	
	−3440	−3400	−3360	−3322	−3284	−3247	−3210	−3174	−3139	−3103	
1·8	·555556	552486	549451	546448	543478	540541	537634	534759	531915	529101	
	−3070	−3035	−3003	−2970	−2937	−2907	−2875	−2844	−2814	−2785	
1·9	·526316	523560	520833	518135	515464	512821	510204	507614	505051	502513	
	−2756	−2727	−2698	−2671	−2643	−2617	−2590	−2563	−2538	−2513	
2·0	·500000	497512	495050	492611	490196	487805	485437	483092	480769	478469	
	−2488	−2462	−2439	−2415	−2391	−2368	−2345	−2323	−2300	−2279	
2·1	·476190	473934	471698	469484	467290	465116	462963	460829	458716	456621	
	−2256	−2236	−2214	−2194	−2174	−2153	−2134	−2113	−2095	−2076	
2·2	·454545	452489	450450	448430	446429	444444	442478	440529	438596	436681	
	−2056	−2039	−2020	−2001	−1985	−1966	−1949	−1933	−1915	−1898	
2·3	·434783	432900	431034	429185	427350	425532	423729	421941	420168	418410	
	−1883	−1866	−1849	−1835	−1818	−1803	−1788	−1773	−1758	−1743	
2·4	·416667	414938	413223	411523	409836	408163	406504	404858	403226	401606	
	−1729	−1715	−1700	−1687	−1673	−1659	−1646	−1632	−1620	−1606	
2·5	·400000	398406	396825	395257	393701	392157	390625	389105	387597	386100	−1544
2·6	·384615	383142	381679	380228	378788	377358	375940	374532	373134	371747	−1430
2·7	·370370	369004	367647	366300	364964	363636	362319	361011	359712	358423	−1327
2·8	·357143	355872	354610	353357	352113	350877	349650	348432	347222	346021	−1236
2·9	·344828	343643	342466	341297	340136	338983	337838	336700	335570	334448	−1153
3·0	·333333	332226	331126	330033	328947	327869	326797	325733	324675	323625	−1079
3·1	·322581	321543	320513	319489	318471	317460	316456	315457	314465	313480	−1011
3·2	·312500	311526	310559	309598	308642	307692	306748	305810	304878	303951	−950
3·3	·303030	302115	301205	300300	299401	298507	297619	296736	295858	294985	−894
3·4	·294118	293255	292398	291545	290698	289855	289017	288184	287356	286533	−843
3·5	·285714	284900	284091	283286	282486	281690	280899	280112	279330	278552	−796
3·6	·277778	277008	276243	275482	274725	273973	273224	272480	271739	271003	−753
3·7	·270270	269542	268817	268097	267380	266667	265957	265252	264550	263852	−713
3·8	·263158	262467	261780	261097	260417	259740	259067	258398	257732	257069	−677
3·9	·256410	255754	255102	254453	253807	253165	252525	251889	251256	250627	−643
4·0	·250000	249377	248756	248139	247525	246914	246305	245700	245098	244499	−611
4·1	·243902	243309	242718	242131	241546	240964	240385	239808	239234	238663	−582
4·2	·238095	237530	236967	236407	235849	235294	234742	234192	233645	233100	−555
4·3	·232558	232019	231481	230947	230415	229885	229358	228833	228311	227790	−530
4·4	·227273	226757	226244	225734	225225	224719	224215	223714	223214	222717	−506
4·5	·222222	221729	221239	220751	220264	219780	219298	218818	218341	217865	−484
4·6	·217391	216920	216450	215983	215517	215054	214592	214133	213675	213220	−463
4·7	·212766	212314	211864	211416	210970	210526	210084	209644	209205	208768	−444
4·8	·208333	207900	207469	207039	206612	206186	205761	205339	204918	204499	−426
4·9	·204082	203666	203252	202840	202429	202020	201613	201207	200803	200401	−409

* Tabular differences up to 2·5, and mean differences thereafter. Differences are *negative*.

TABLE XXIX. RECIPROCALS—*continued*

	0	1	2	3	4	5	6	7	8	9	Diffs.*
5·0	·200000	199601	199203	198807	198413	198020	197628	197239	196850	196464	−393
5·1	·196078	195695	195312	194932	194553	194175	193798	193424	193050	192678	−378
5·2	·192308	191939	191571	191205	190840	190476	190114	189753	189394	189036	−364
5·3	·188679	188324	187970	187617	187266	186916	186567	186220	185874	185529	−350
5·4	·185185	184843	184502	184162	183824	183486	183150	182815	182482	182149	−337
5·5	·181818	181488	181159	180832	180505	180180	179856	179533	179211	178891	−325
5·6	·178571	178253	177936	177620	177305	176991	176678	176367	176056	175747	−314
5·7	·175439	175131	174825	174520	174216	173913	173611	173310	173010	172712	−303
5·8	·172414	172117	171821	171527	171233	170940	170648	170358	170068	169779	−293
5·9	·169492	169205	168919	168634	168350	168067	167785	167504	167224	166945	−283
6·0	·166667	166389	166113	165837	165563	165289	165017	164745	164474	164204	−274
6·1	·163934	163666	163399	163132	162866	162602	162338	162075	161812	161551	−265
6·2	·161290	161031	160772	160514	160256	160000	159744	159490	159236	158983	−256
6·3	·158730	158479	158228	157978	157729	157480	157233	156986	156740	156495	−248
6·4	·156250	156006	155763	155521	155280	155039	154799	154560	154321	154083	−241
6·5	·153846	153610	153374	153139	152905	152672	152439	152207	151976	151745	−233
6·6	·151515	151286	151057	150830	150602	150376	150150	149925	149701	149477	−226
6·7	·149254	149031	148810	148588	148368	148148	147929	147710	147493	147275	−220
6·8	·147059	146843	146628	146413	146199	145985	145773	145560	145349	145138	−213
6·9	·144928	144718	144509	144300	144092	143885	143678	143472	143266	143062	−207
7·0	·142857	142653	142450	142248	142045	141844	141643	141443	141243	141044	−201
7·1	·140845	140647	140449	140252	140056	139860	139665	139470	139276	139082	−196
7·2	·138889	138696	138504	138313	138122	137931	137741	137552	137363	137174	−191
7·3	·136986	136799	136612	136426	136240	136054	135870	135685	135501	135318	−185
7·4	·135135	134953	134771	134590	134409	134228	134048	133869	133690	133511	−180
7·5	·133333	133156	132979	132802	132626	132450	132275	132100	131926	131752	−176
7·6	·131579	131406	131234	131062	130890	130719	130548	130378	130208	130039	−171
7·7	·129870	129702	129534	129366	129199	129032	128866	128700	128535	128370	−167
7·8	·128205	128041	127877	127714	127551	127389	127226	127065	126904	126743	−162
7·9	·126582	126422	126263	126103	125945	125786	125628	125471	125313	125156	−158
8·0	·125000	124844	124688	124533	124378	124224	124069	123916	123762	123609	−155
8·1	·123457	123305	123153	123001	122850	122699	122549	122399	122249	122100	−151
8·2	·121951	121803	121655	121507	121359	121212	121065	120919	120773	120627	−147
8·3	·120482	120337	120192	120048	119904	119760	119617	119474	119332	119190	−144
8·4	·119048	118906	118765	118624	118483	118343	118203	118064	117925	117786	−140
8·5	·117647	117509	117371	117233	117096	116959	116822	116686	116550	116414	−137
8·6	·116279	116144	116009	115875	115741	115607	115473	115340	115207	115075	−134
8·7	·114943	114811	114679	114548	114416	114286	114155	114025	113895	113766	−131
8·8	·113636	113507	113379	113250	113122	112994	112867	112740	112613	112486	−128
8·9	·112360	112233	112108	111982	111857	111732	111607	111483	111359	111235	−125
9·0	·111111	110988	110865	110742	110619	110497	110375	110254	110132	110011	−122
9·1	·109890	109769	109649	109529	109409	109290	109170	109051	108932	108814	−120
9·2	·108696	108578	108460	108342	108225	108108	107991	107875	107759	107643	−117
9·3	·107527	107411	107296	107181	107066	106952	106838	106724	106610	106496	−115
9·4	·106383	106270	106157	106045	105932	105820	105708	105597	105485	105374	−112
9·5	·105263	105152	105042	104932	104822	104712	104603	104493	104384	104275	−110
9·6	·104167	104058	103950	103842	103734	103627	103520	103413	103306	103199	−108
9·7	·103093	102987	102881	102775	102669	102564	102459	102354	102249	102145	−105
9·8	·102041	101937	101833	101729	101626	101523	101420	101317	101215	101112	−103
9·9	·101010	100908	100806	100705	100604	100503	100402	100301	100200	100100	−101

Thus 1/4·214 = 0·237530−0·000222 = 0·237308, since 0·4 × 555 = 222 (correct value 0·237304).

Table XXX. Factorials

No.	Factorial.	Logarithm.	No.	Factorial.	Logarithm.	No.	Factorial.	Logarithm.
1	1	0·000 0000	51	1·55112	66·190 6450	101	9·42595	159·974 3250
2	2	0·301 0300	52	8·06582	67·906 6484	102	9·61447	161·982 9252
3	6	0·778 1513	53	4·27488	69·630 9243	103	9·90290	163·995 7624
4	24	1·380 2112	54	2·30844	71·363 3180	104	1·02990	166·012 7958
5	120	2·079 1812	55	1·26964	73·103 6807	105	1·08140	168·033 9851
6	720	2·857 3325	56	7·10999	74·851 8687	106	1·14628	170·059 2909
7	5040	3·702 4305	57	4·05269	76·607 7436	107	1·22652	172·088 6747
8	40320	4·605 5205	58	2·35056	78·371 1716	108	1·32464	174·122 0985
9	362880	5·559 7630	59	1·38683	80·142 0236	109	1·44386	176·159 5250
10	3·62880	6·559 7630	60	8·32099	81·920 1748	110	1·58825	178·200 9176
11	3·99168	7·601 1557	61	5·07580	83·705 5047	111	1·76295	180·246 2406
12	4·79002	8·680 3370	62	3·14700	85·497 8964	112	1·97451	182·295 4586
13	6·22702	9·794 2803	63	1·98261	87·297 2369	113	2·23119	184·348 5371
14	8·71783	10·940 4084	64	1·26887	89·103 4169	114	2·54356	186·405 4419
15	1·30767	12·116 4996	65	8·24765	90·916 3303	115	2·92509	188·466 1398
16	2·09228	13·320 6196	66	5·44345	92·735 8742	116	3·39311	190·530 5978
17	3·55687	14·551 0685	67	3·64711	94·561 9490	117	3·96994	192·598 7836
18	6·40237	15·806 3410	68	2·48004	96·394 4579	118	4·68453	194·670 6656
19	1·21645	17·085 0946	69	1·71122	98·233 3070	119	5·57459	196·746 2126
20	2·43290	18·386 1246	70	1·19786	100·078 4050	120	6·68950	198·825 3938
21	5·10909	19·708 3439	71	8·50479	101·929 6634	121	8·09430	200·908 1792
22	1·12400	21·050 7666	72	6·12345	103·786 9959	122	9·87504	202·994 5390
23	2·58520	22·412 4944	73	4·47012	105·650 3187	123	1·21463	205·084 4442
24	6·20448	23·792 7057	74	3·30789	107·519 5505	124	1·50614	207·177 8658
25	1·55112	25·190 6457	75	2·48091	109·394 6117	125	1·88268	209·274 7759
26	4·03291	26·605 6190	76	1·88549	111·275 4253	126	2·37217	211·375 1464
27	1·08889	28·036 9828	77	1·45183	113·161 9160	127	3·01266	213·478 9501
28	3·04888	29·484 1408	78	1·13243	115·054 0106	128	3·85620	215·586 1601
29	8·84176	30·946 5388	79	8·94618	116·951 6377	129	4·97450	217·696 7498
30	2·65253	32·423 6601	80	7·15695	118·854 7277	130	6·46686	219·810 6932
31	8·22284	33·915 0218	81	5·79713	120·763 2127	131	8·47158	221·927 9645
32	2·63131	35·420 1717	82	4·75364	122·677 0266	132	1·11825	224·048 5384
33	8·68332	36·938 6857	83	3·94552	124·596 1047	133	1·48727	226·172 3900
34	2·95233	38·470 1646	84	3·31424	126·520 3840	134	1·99294	228·299 4948
35	1·03331	40·014 2326	85	2·81710	128·449 8029	135	2·69047	230·429 8286
36	3·71993	41·570 5351	86	2·42271	130·384 3013	136	3·65904	232·563 3675
37	1·37638	43·138 7369	87	2·10776	132·323 8206	137	5·01289	234·700 0881
38	5·23023	44·718 5205	88	1·85483	134·268 3033	138	6·91779	236·839 9672
39	2·03979	46·309 5851	89	1·65080	136·217 6933	139	9·61572	238·982 9820
40	8·15915	47·911 6451	90	1·48572	138·171 9358	140	1·34620	241·129 1100
41	3·34525	49·524 4289	91	1·35200	140·130 9772	141	1·89814	243·278 3291
42	1·40501	51·147 6782	92	1·24384	142·094 7650	142	2·69536	245·430 6174
43	6·04153	52·781 1467	93	1·15677	144·063 2480	143	3·85437	247·585 9535
44	2·65827	54·424 5993	94	1·08737	146·036 3758	144	5·55029	249·744 3160
45	1·19622	56·077 8119	95	1·03300	148·014 0994	145	8·04793	251·905 6840
46	5·50262	57·740 5697	96	9·91678	149·996 3707	146	1·17500	254·070 0368
47	2·58623	59·412 6676	97	9·61928	151·983 1424	147	1·72725	256·237 3542
48	1·24139	61·093 9088	98	9·42689	153·974 3685	148	2·55632	258·407 6159
49	6·08282	62·784 1049	99	9·33262	155·970 0037	149	3·80892	260·580 8022
50	3·04141	64·483 0749	100	9·33262	157·970 0037	150	5·71338	262·756 8934

The power of 10 by which to multiply the factorial is given by the whole number of the logarithm.

No.	Factorial.	Logarithm.	No.	Factorial.	Logarithm.	No.	Factorial.	Logarithm.
151	8·62721	264·935 8704	201	1·58520	377·200 0847	251	8·11447	494·909 2601
152	1·31134	267·117 7139	202	3·20211	379·505 4361	252	2·04485	497·310 6607
153	2·00634	269·302 4054	203	6·50028	381·812 9321	253	5·17346	499·713 7812
154	3·08977	271·489 9261	204	1·32606	384·122 5623	254	1·31406	502·118 6149
155	4·78914	273·680 2578	205	2·71842	386·434 3161	255	3·35085	504·525 1551
156	7·47106	275·873 3824	206	5·59994	388·748 1834	256	8·57818	506·933 3950
157	1·17296	278·069 2820	207	1·15919	391·064 1537	257	2·20459	509·343 3282
158	1·85327	280·267 9391	208	2·41111	393·382 2170	258	5·68785	511·754 9479
159	2·94670	282·469 3363	209	5·03922	395·702 3633	259	1·47315	514·168 2476
160	4·71472	284·673 4562	210	1·05824	398·024 5826	260	3·83020	516·583 2210
161	7·59070	286·880 2821	211	2·23288	400·348 8651	261	9·99681	518·999 8615
162	1·22969	289·089 7971	212	4·73370	402·675 2009	262	2·61916	521·418 1628
163	2·00440	291·301 9847	213	1·00828	405·003 5805	263	6·88840	523·838 1185
164	3·28722	293·516 8286	214	2·15772	407·333 9943	264	1·81854	526·259 7225
165	5·42391	295·734 3125	215	4·63909	409·666 4328	265	4·81913	528·682 9683
166	9·00369	297·954 4206	216	1·00204	412·000 8865	266	1·28189	531·107 8500
167	1·50362	300·177 1371	217	2·17443	414·337 3463	267	3·42264	533·534 3612
168	2·52608	302·402 4464	218	4·74027	416·675 8027	268	9·17268	535·962 4960
169	4·26907	304·630 3331	219	1·03812	419·016 2469	269	2·46745	538·392 2483
170	7·25742	306·860 7820	220	2·28386	421·358 6695	270	6·66211	540·823 6121
171	1·24102	309·093 7781	221	5·04733	423·703 0618	271	1·80543	543·256 5814
172	2·13455	311·329 3066	222	1·12051	426·049 4148	272	4·91078	545·691 1503
173	3·69277	313·567 3527	223	2·49873	428·397 7197	273	1·34064	548·127 3129
174	6·42543	315·807 9019	224	5·59716	430·747 9677	274	3·67336	550·565 0635
175	1·12445	318·050 9400	225	1·25936	433·100 1502	275	1·01017	553·004 3962
176	1·97903	320·296 4526	226	2·84616	435·454 2586	276	2·78808	555·445 3052
177	3·50289	322·544 4259	227	6·46077	437·810 2845	277	7·72298	557·887 7850
178	6·23514	324·794 8459	228	1·47306	440·168 2193	278	2·14699	560·331 8298
179	1·11609	327·047 6989	229	3·37330	442·528 0548	279	5·99010	562·777 4340
180	2·00896	329·302 9714	230	7·75859	444·889 7827	280	1·67723	565·224 5920
181	3·63622	331·560 6500	231	1·79223	447·253 3946	281	4·71301	567·673 2984
182	6·61792	333·820 7214	232	4·15798	449·618 8826	282	1·32907	570·123 5475
183	1·21108	336·083 1725	233	9·68810	451·986 2385	283	3·76126	572·575 3339
184	2·22839	338·347 9903	234	2·26702	454·355 4544	284	1·06820	575·028 6523
185	4·12251	340·615 1620	235	5·32749	456·726 5223	285	3·04437	577·483 4971
186	7·66787	342·884 6750	236	1·25729	459·099 4343	286	8·70689	579·939 8631
187	1·43389	345·156 5166	237	2·97977	461·474 1826	287	2·49888	582·397 7450
188	2·69572	347·430 6744	238	7·09185	463·850 7596	288	7·19677	584·857 1375
189	5·09491	349·707 1362	239	1·69495	466·229 1575	289	2·07987	587·318 0354
190	9·68032	351·985 8898	240	4·06789	468·609 3687	290	6·03161	589·780 4334
191	1·84894	354·266 9232	241	9·80360	470·991 3857	291	1·75520	592·244 3264
192	3·54997	356·550 2244	242	2·37247	473·375 2011	292	5·12518	594·709 7092
193	6·85144	358·835 7817	243	5·76511	475·760 8074	293	1·50168	597·176 5768
194	1·32918	361·123 5835	244	1·40669	478·148 1972	294	4·41493	599·644 9242
195	2·59190	363·413 6181	245	3·44638	480·537 3633	295	1·30241	602·114 7462
196	5·08012	365·705 8742	246	8·47810	482·928 2984	296	3·85512	604·586 0379
197	1·00078	368·000 3404	247	2·09409	485·320 9954	297	1·14497	607·058 7943
198	1·98155	370·297 0056	248	5·19334	487·715 4470	298	3·41201	609·533 0106
199	3·94329	372·595 8586	249	1·29314	490·111 6464	299	1·02019	612·008 6818
200	7·88658	374·896 8886	250	3·23286	492·509 5864	300	3·06058	614·485 8030

For higher values use : $\log x! = (x+\frac{1}{2}) \log x + \frac{1}{2} \log 2\pi - \left(x - \frac{1}{12\,x}\right) \log e.$

Table XXXI. Natural Sines

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′
0	·00000	00175	00349	00524	00698	00873	01047	01222	01396	01571	29	58	87
1	·01745	01920	02094	02269	02443	02618	02792	02967	03141	03316	29	58	87
2	·03490	03664	03839	04013	04188	04362	04536	04711	04885	05059	29	58	87
3	·05234	05408	05582	05756	05931	06105	06279	06453	06627	06802	29	58	87
4	·06976	07150	07324	07498	07672	07846	08020	08194	08368	08542	29	58	87
5	·08716	08889	09063	09237	09411	09585	09758	09932	10106	10279	29	58	87
6	·10453	10626	10800	10973	11147	11320	11494	11667	11840	12014	29	58	87
7	·12187	12360	12533	12706	12880	13053	13226	13399	13572	13744	29	58	86
8	·13917	14090	14263	14436	14608	14781	14954	15126	15299	15471	29	58	86
9	·15643	15816	15988	16160	16333	16505	16677	16849	17021	17193	29	57	86
10	·17365	17537	17708	17880	18052	18224	18395	18567	18738	18910	29	57	86
11	·19081	19252	19423	19595	19766	19937	20108	20279	20450	20620	28	57	86
12	·20791	20962	21132	21303	21474	21644	21814	21985	22155	22325	28	57	85
13	·22495	22665	22835	23005	23175	23345	23514	23684	23853	24023	28	57	85
14	·24192	24362	24531	24700	24869	25038	25207	25376	25545	25713	28	56	85
15	·25882	26050	26219	26387	26556	26724	26892	27060	27228	27396	28	56	84
16	·27564	27731	27899	28067	28234	28402	28569	28736	28903	29070	28	56	84
17	·29237	29404	29571	29737	29904	30071	30237	30403	30570	30736	28	56	83
18	·30902	31068	31233	31399	31565	31730	31896	32061	32227	32392	28	55	83
19	·32557	32722	32887	33051	33216	33381	33545	33710	33874	34038	27	55	82
20	·34202	34366	34530	34694	34857	35021	35184	35347	35511	35674	27	55	82
21	·35837	36000	36162	36325	36488	36650	36812	36975	37137	37299	27	54	81
22	·37461	37622	37784	37946	38107	38268	38430	38591	38752	38912	27	54	81
23	·39073	39234	39394	39555	39715	39875	40035	40195	40355	40514	27	53	80
24	·40674	40833	40992	41151	41310	41469	41628	41787	41945	42104	26	53	79
25	·42262	42420	42578	42736	42894	43051	43209	43366	43523	43680	26	53	79
26	·43837	43994	44151	44307	44464	44620	44776	44932	45088	45243	26	52	78
27	·45399	45554	45710	45865	46020	46175	46330	46484	46639	46793	26	52	77
28	·46947	47101	47255	47409	47562	47716	47869	48022	48175	48328	26	51	77
29	·48481	48634	48786	48938	49090	49242	49394	49546	49697	49849	25	51	76
30	·50000	50151	50302	50453	50603	50754	50904	51054	51204	51354	25	50	75
31	·51504	51653	51803	51952	52101	52250	52399	52547	52696	52844	25	50	74
32	·52992	53140	53288	53435	53583	53730	53877	54024	54171	54317	25	49	74
33	·54464	54610	54756	54902	55048	55194	55339	55484	55630	55775	24	49	73
34	·55919	56064	56208	56353	56497	56641	56784	56928	57071	57215	24	48	72
35	·57358	57501	57643	57786	57928	58070	58212	58354	58496	58637	24	47	71
36	·58779	58920	59061	59201	59342	59482	59622	59763	59902	60042	23	47	70
37	·60182	60321	60460	60599	60738	60876	61015	61153	61291	61429	23	46	69
38	·61566	61704	61841	61978	62115	62251	62388	62524	62660	62796	23	46	68
39	·62932	63068	63203	63338	63473	63608	63742	63877	64011	64145	22	45	67
40	·64279	64412	64546	64679	64812	64945	65077	65210	65342	65474	22	44	66
41	·65606	65738	65869	66000	66131	66262	66393	66523	66653	66783	22	44	65
42	·66913	67043	67172	67301	67430	67559	67688	67816	67944	68072	21	43	64
43	·68200	68327	68455	68582	68709	68835	68962	69088	69214	69340	21	42	63
44	·69466	69591	69717	69842	69966	70091	70215	70339	70463	70587	21	42	62

TABLE XXXI. NATURAL SINES—*continued*

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′
45	·70711	70834	70957	71080	71203	71325	71447	71569	71691	71813	20	41	61
46	·71934	72055	72176	72297	72417	72537	72657	72777	72897	73016	20	40	60
47	·73135	73254	73373	73491	73610	73728	73846	73963	74080	74198	20	39	59
48	·74314	74431	74548	74664	74780	74896	75011	75126	75241	75356	19	39	58
49	·75471	75585	75700	75813	75927	76041	76154	76267	76380	76492	19	38	57
50	·76604	76717	76828	76940	77051	77162	77273	77384	77494	77605	19	37	56
51	·77715	77824	77934	78043	78152	78261	78369	78478	78586	78694	18	36	54
52	·78801	78908	79016	79122	79229	79335	79441	79547	79653	79758	18	35	53
53	·79864	79968	80073	80178	80282	80386	80489	80593	80696	80799	17	35	52
54	·80902	81004	81106	81208	81310	81412	81513	81614	81714	81815	17	34	51
55	·81915	82015	82115	82214	82314	82413	82511	82610	82708	82806	16	33	50
56	·82904	83001	83098	83195	83292	83389	83485	83581	83676	83772	16	32	48
57	·83867	83962	84057	84151	84245	84339	84433	84526	84619	84712	16	31	47
58	·84805	84897	84989	85081	85173	85264	85355	85446	85536	85627	15	30	46
59	·85717	85806	85896	85985	86074	86163	86251	86340	86427	86515	15	30	44
60	·86603	86690	86777	86863	86949	87036	87121	87207	87292	87377	14	29	43
61	·87462	87546	87631	87715	87798	87882	87965	88048	88130	88213	14	28	42
62	·88295	88377	88458	88539	88620	88701	88782	88862	88942	89021	13	27	40
63	·89101	89180	89259	89337	89415	89493	89571	89649	89726	89803	13	26	39
64	·89879	89956	90032	90108	90183	90259	90334	90408	90483	90557	13	25	38
65	·90631	90704	90778	90851	90924	90996	91068	91140	91212	91283	12	24	36
66	·91355	91425	91496	91566	91636	91706	91775	91845	91914	91982	12	23	35
67	·92050	92119	92186	92254	92321	92388	92455	92521	92587	92653	11	22	34
68	·92718	92784	92849	92913	92978	93042	93106	93169	93232	93295	11	21	32
69	·93358	93420	93483	93544	93606	93667	93728	93789	93849	93909	10	20	31
70	·93969	94029	94088	94147	94206	94264	94322	94380	94438	94495	10	19	29
71	·94552	94609	94665	94721	94777	94832	94888	94943	94997	95052	9	19	28
72	·95106	95159	95213	95266	95319	95372	95424	95476	95528	95579	9	18	26
73	·95630	95681	95732	95782	95832	95882	95931	95981	96029	96078	8	17	25
74	·96126	96174	96222	96269	96316	96363	96410	96456	96502	96547	8	16	23
75	·96593	96638	96682	96727	96771	96815	96858	96902	96945	96987	7	15	22
76	·97030	97072	97113	97155	97196	97237	97278	97318	97358	97398	7	14	20
77	·97437	97476	97515	97553	97592	97630	97667	97705	97742	97778	6	13	19
78	·97815	97851	97887	97922	97958	97992	98027	98061	98096	98129	6	12	17
79	·98163	98196	98229	98261	98294	98325	98357	98389	98420	98450	5	11	16
80	·98481	98511	98541	98570	98600	98629	98657	98686	98714	98741	5	10	14
81	·98769	98796	98823	98849	98876	98902	98927	98953	98978	99002	4	9	13
82	·99027	99051	99075	99098	99122	99144	99167	99189	99211	99233	4	8	11
83	·99255	99276	99297	99317	99337	99357	99377	99396	99415	99434	3	7	10
84	·99452	99470	99488	99506	99523	99540	99556	99572	99588	99604	3	6	8
85	·99619	99635	99649	99664	99678	99692	99705	99719	99731	99744	2	5	7
86	·99756	99768	99780	99792	99803	99813	99824	99834	99844	99854	2	4	5
87	·99863	99872	99881	99889	99897	99905	99912	99919	99926	99933	1	3	4
88	·99939	99945	99951	99956	99961	99966	99970	99974	99978	99982	1	2	2
89	·99985	99988	99990	99993	99995	99996	99998	99999	99999	100000	0	1	1

TABLE XXXII. NATURAL TANGENTS

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′
0	0·00000	·00175	·00349	·00524	·00698	·00873	·01047	·01222	·01396	·01571	29	58	87
1	0·01746	·01920	·02095	·02269	·02444	·02619	·02793	·02968	·03143	·03317	29	58	87
2	0·03492	·03667	·03842	·04016	·04191	·04366	·04541	·04716	·04891	·05066	29	58	87
3	0·05241	·05416	·05591	·05766	·05941	·06116	·06291	·06467	·06642	·06817	29	58	88
4	0·06993	·07168	·07344	·07519	·07695	·07870	·08046	·08221	·08397	·08573	29	59	88
5	0·08749	·08925	·09101	·09277	·09453	·09629	·09805	·09981	·10158	·10334	29	59	88
6	0·10510	·10687	·10863	·11040	·11217	·11394	·11570	·11747	·11924	·12101	29	59	88
7	0·12278	·12456	·12633	·12810	·12988	·13165	·13343	·13521	·13698	·13876	30	59	89
8	0·14054	·14232	·14410	·14588	·14767	·14945	·15124	·15302	·15481	·15660	30	59	89
9	0·15838	·16017	·16196	·16376	·16555	·16734	·16914	·17093	·17273	·17453	30	60	90
10	0·17633	·17813	·17993	·18173	·18353	·18534	·18714	·18895	·19076	·19257	30	60	90
11	0·19438	·19619	·19801	·19982	·20164	·20345	·20527	·20709	·20891	·21073	30	61	91
12	0·21256	·21438	·21621	·21804	·21986	·22169	·22353	·22536	·22719	·22903	30	61	92
13	0·23087	·23271	·23455	·23639	·23823	·24008	·24193	·24377	·24562	·24747	31	61	92
14	0·24933	·25118	·25304	·25490	·25676	·25862	·26048	·26235	·26421	·26608	31	62	93
15	0·26795	·26982	·27169	·27357	·27545	·27732	·27921	·28109	·28297	·28486	31	63	94
16	0·28675	·28864	·29053	·29242	·29432	·29621	·29811	·30001	·30192	·30382	32	63	95
17	0·30573	·30764	·30955	·31147	·31338	·31530	·31722	·31914	·32106	·32299	32	64	96
18	0·32492	·32685	·32878	·33072	·33266	·33460	·33654	·33848	·34043	·34238	32	65	97
19	0·34433	·34628	·34824	·35020	·35216	·35412	·35608	·35805	·36002	·36199	33	65	98
20	0·36397	·36595	·36793	·36991	·37190	·37388	·37588	·37787	·37986	·38186	33	66	99
21	0·38386	·38587	·38787	·38988	·39190	·39391	·39593	·39795	·39997	·40200	34	67	101
22	0·40403	·40606	·40809	·41013	·41217	·41421	·41626	·41831	·42036	·42242	34	68	102
23	0·42447	·42654	·42860	·43067	·43274	·43481	·43689	·43897	·44105	·44314	35	69	104
24	0·44523	·44732	·44942	·45152	·45362	·45573	·45784	·45995	·46206	·46418	35	70	105
25	0·46631	·46843	·47056	·47270	·47483	·47698	·47912	·48127	·48342	·48557	36	71	107
26	0·48773	·48989	·49206	·49423	·49640	·49858	·50076	·50295	·50514	·50733	36	73	109
27	0·50953	·51173	·51393	·51614	·51835	·52057	·52279	·52501	·52724	·52947	37	74	111
28	0·53171	·53395	·53620	·53844	·54070	·54296	·54522	·54748	·54975	·55203	38	75	113
29	0·55431	·55659	·55888	·56117	·56347	·56577	·56808	·57039	·57271	·57503	38	77	115
30	0·57735	·57968	·58201	·58435	·58670	·58905	·59140	·59376	·59612	·59849	39	78	117
31	0·60086	·60324	·60562	·60801	·61040	·61280	·61520	·61761	·62003	·62245	40	80	120
32	0·62487	·62730	·62973	·63217	·63462	·63707	·63953	·64199	·64446	·64693	41	82	123
33	0·64941	·65189	·65438	·65688	·65938	·66189	·66440	·66692	·66944	·67197	42	84	125
34	0·67451	·67705	·67960	·68215	·68471	·68728	·68985	·69243	·69502	·69761	43	86	128
35	0·70021	·70281	·70542	·70804	·71066	·71329	·71593	·71857	·72122	·72388	44	88	131
36	0·72654	·72921	·73189	·73457	·73726	·73996	·74267	·74538	·74810	·75082	45	90	135
37	0·75355	·75629	·75904	·76180	·76456	·76733	·77010	·77289	·77568	·77848	46	92	139
38	0·78129	·78410	·78692	·78975	·79259	·79544	·79829	·80115	·80402	·80690	47	95	142
39	0·80978	·81268	·81558	·81849	·82141	·82434	·82727	·83022	·83317	·83613	49	98	146
40	0·83910	·84208	·84507	·84806	·85107	·85408	·85710	·86014	·86318	·86623	50	100	151
41	0·86929	·87236	·87543	·87852	·88162	·88473	·88784	·89097	·89410	·89725	52	104	155
42	0·90040	·90357	·90674	·90993	·91313	·91633	·91955	·92277	·92601	·92926	53	107	160
43	0·93252	·93578	·93906	·94235	·94565	·94896	·95229	·95562	·95897	·96232	55	110	166
44	0·96569	·96907	·97246	·97586	·97927	·98270	·98613	·98958	·99304	·99652	57	114	171

TABLE XXXII. NATURAL TANGENTS—*continued*

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′
45	1·00000	·00350	·00701	·01053	·01406	·01761	·02117	·02474	·02832	·03192	59	118	177
46	1·03553	·03915	·04279	·04644	·05010	·05378	·05747	·06117	·06489	·06862	61	123	184
47	1·07237	·07613	·07990	·08369	·08749	·09131	·09514	·09899	·10285	·10672	64	127	191
48	1·11061	·11452	·11844	·12238	·12633	·13029	·13428	·13828	·14229	·14632	66	132	198
49	1·15037	·15443	·15851	·16261	·16672	·17085	·17500	·17916	·18334	·18754	69	138	206
50	1·19175	·19599	·20024	·20451	·20879	·21310	·21742	·22176	·22612	·23050	72	144	215
51	1·23490	·23931	·24375	·24820	·25268	·25717	·26169	·26622	·27077	·27535	75	150	225
52	1·27994	·28456	·28919	·29385	·29853	·30323	·30795	·31269	·31745	·32224	78	157	235
53	1·32704	·33187	·33673	·34160	·34650	·35142	·35637	·36134	·36633	·37134	82	164	246
54	1·37638	·38145	·38653	·39165	·39679	·40195	·40714	·41235	·41759	·42286	86	172	258
55	1·42815	·43347	·43881	·44418	·44958	·45501	·46046	·46595	·47146	·47699	90	181	271
56	1·48256	·48816	·49378	·49944	·50512	·51084	·51658	·52235	·52816	·53400	95	191	286
57	1·53986	·54576	·55170	·55766	·56366	·56969	·57575	·58184	·58797	·59414	100	201	302
58	1·60033	·60657	61283	·61914	·62548	·63185	·63826	·64471	·65120	·65772	106	213	319
59	1·66428	·67088	·67752	·68419	·69091	·69766	·70446	·71129	·71817	·72509	113	225	338
60	1·73205	·73905	·74610	·75319	·76032	·76749	·77471	·78198	·78929	·79665	—	—	—

$(90° - \theta) \tan \theta$

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	tan θ
60	51·962	51·998	52·034	52·070	52·105	52·141	52·177	52·212	52·247	52·282	6	12	18	1·73
61	52·317	52·352	52·387	52·422	52·456	52·490	52·525	52·559	52·593	52·627	6	11	17	1·80
62	52·660	52·694	52·727	52·761	52·794	52·827	52·860	52·893	52·925	52·958	6	11	17	1·88
63	52·990	53·023	53·055	53·087	53·119	53·151	53·182	53·214	53·245	53·277	5	11	16	1·96
64	53·308	53·339	53·370	53·401	53·431	53·462	53·492	53·523	53·553	53·583	5	10	15	2·05
65	53·613	53·642	53·672	53·702	53·731	53·760	53·790	53·819	53·847	53·876	5	10	15	2·14
66	53·905	53·933	53·962	53·990	54·018	54·046	54·074	54·102	54·130	54·157	5	9	14	2·25
67	54·185	54·212	54·239	54·266	54·293	54·320	54·346	54·373	54·399	54·426	4	9	13	2·36
68	54·452	54·478	54·504	54·530	54·555	54·581	54·606	54·632	54·657	54·682	4	9	13	2·48
69	54·707	54·732	54·756	54·781	54·805	54·830	54·854	54·878	54·902	54·926	4	8	12	2·61
70	54·950	54·973	54·997	55·020	55·043	55·066	55·089	55·112	55·135	55·158	4	8	12	2·75
71	55·180	55·202	55·225	55·247	55·269	55·291	55·312	55·334	55·356	55·377	4	7	11	2·90
72	55·398	55·419	55·441	55·461	55·482	55·503	55·523	55·544	55·564	55·584	3	7	10	3·08
73	55·604	55·624	55·644	55·664	55·684	55·703	55·722	55·742	55·761	55·780	3	7	10	3·27
74	55·799	55·817	55·836	55·855	55·873	55·891	55·909	55·927	55·945	55·963	3	6	9	3·49
75	55·981	55·998	56·016	56·033	56·050	56·067	56·084	56·101	56·118	56·134	3	6	8	3·73
76	56·151	56·167	56·184	56·200	56·216	56·232	56·247	56·263	56·278	56·294	3	5	8	4·01
77	56·309	56·324	56·339	56·354	56·369	56·384	56·398	56·413	56·427	56·441	2	5	7	4·33
78	56·456	56·470	56·483	56·497	56·511	56·524	56·538	56·551	56·564	56·577	2	4	7	4·70
79	56·590	56·603	56·616	56·628	56·641	56·653	56·665	56·677	56·689	56·701	2	4	6	5·14
80	56·713	56·724	56·736	56·747	56·759	56·770	56·781	56·792	56·803	56·813	2	4	6	5·67
81	56·824	56·834	56·845	56·855	56·865	56·875	56·885	56·894	56·904	56·914	2	3	5	6·31
82	56·923	56·932	56·941	56·950	56·959	56·968	56·977	56·985	56·994	57·002	1	3	4	7·12
83	57·010	57·019	57·027	57·034	57·042	57·050	57·057	57·065	57·072	57·079	1	3	4	8·14
84	57·086	57·093	57·100	57·107	57·113	57·120	57·126	57·132	57·138	57·144	1	2	3	9·51
85	57·150	57·156	57·162	57·167	57·173	57·178	57·183	57·188	57·193	57·198	1	2	3	11·4
86	57·203	57·207	57·212	57·216	57·220	57·224	57·229	57·232	57·236	57·240	1	1	2	14·3
87	57·243	57·247	57·250	57·253	57·256	57·259	57·262	57·265	57·268	57·270	0	1	2	19·1
88	57·273	57·275	57·277	57·279	57·281	57·283	57·284	57·286	57·287	57·289	0	1	1	28·6
89	57·290	57·291	57·292	57·293	57·294	57·294	57·295	57·295	57·296	57·296	0	0	0	57·3

TABLE XXXIII. RANDOM NUMBERS (I)

03 47 43 73 86	36 96 47 36 61	46 98 63 71 62	33 26 16 80 45	60 11 14 10 95
97 74 24 67 62	42 81 14 57 20	42 53 32 37 32	27 07 36 07 51	24 51 79 89 73
16 76 62 27 66	56 50 26 71 07	32 90 79 78 53	13 55 38 58 59	88 97 54 14 10
12 56 85 99 26	96 96 68 27 31	05 03 72 93 15	57 12 10 14 21	88 26 49 81 76
55 59 56 35 64	38 54 82 46 22	31 62 43 09 90	06 18 44 32 53	23 83 01 30 30
16 22 77 94 39	49 54 43 54 82	17 37 93 23 78	87 35 20 96 43	84 26 34 91 64
84 42 17 53 31	57 24 55 06 88	77 04 74 47 67	21 76 33 50 25	83 92 12 06 76
63 01 63 78 59	16 95 55 67 19	98 10 50 71 75	12 86 73 58 07	44 39 52 38 79
33 21 12 34 29	78 64 56 07 82	52 42 07 44 38	15 51 00 13 42	99 66 02 79 54
57 60 86 32 44	09 47 27 96 54	49 17 46 09 62	90 52 84 77 27	08 02 73 43 28
18 18 07 92 46	44 17 16 58 09	79 83 86 19 62	06 76 50 03 10	55 23 64 05 05
26 62 38 97 75	84 16 07 44 99	83 11 46 32 24	20 14 85 88 45	10 93 72 88 71
23 42 40 64 74	82 97 77 77 81	07 45 32 14 08	32 98 94 07 72	93 85 79 10 75
52 36 28 19 95	50 92 26 11 97	00 56 76 31 38	80 22 02 53 53	86 60 42 04 53
37 85 94 35 12	83 39 50 08 30	42 34 07 96 88	54 42 06 87 98	35 85 29 48 39
70 29 17 12 13	40 33 20 38 26	13 89 51 03 74	17 76 37 13 04	07 74 21 19 30
56 62 18 37 35	96 83 50 87 75	97 12 25 93 47	70 33 24 03 54	97 77 46 44 80
99 49 57 22 77	88 42 95 45 72	16 64 36 16 00	04 43 18 66 79	94 77 24 21 90
16 08 15 04 72	33 27 14 34 09	45 59 34 68 49	12 72 07 34 45	99 27 72 95 14
31 16 93 32 43	50 27 89 87 19	20 15 37 00 49	52 85 66 60 44	38 68 88 11 80
68 34 30 13 70	55 74 30 77 40	44 22 78 84 26	04 33 46 09 52	68 07 97 06 57
74 57 25 65 76	59 29 97 68 60	71 91 38 67 54	13 58 18 24 76	15 54 55 95 52
27 42 37 86 53	48 55 90 65 72	96 57 69 36 10	96 46 92 42 45	97 60 49 04 91
00 39 68 29 61	66 37 32 20 30	77 84 57 03 29	10 45 65 04 26	11 04 96 67 24
29 94 98 94 24	68 49 69 10 82	53 75 91 93 30	34 25 20 57 27	40 48 73 51 92
16 90 82 66 59	83 62 64 11 12	67 19 00 71 74	60 47 21 29 68	02 02 37 03 31
11 27 94 75 06	06 09 19 74 66	02 94 37 34 02	76 70 90 30 86	38 45 94 30 38
35 24 10 16 20	33 32 51 26 38	79 78 45 04 91	16 92 53 56 16	02 75 50 95 98
38 23 16 86 38	42 38 97 01 50	87 75 66 81 41	40 01 74 91 62	48 51 84 08 32
31 96 25 91 47	96 44 33 49 13	34 86 82 53 91	00 52 43 48 85	27 55 26 89 62
66 67 40 67 14	64 05 71 95 86	11 05 65 09 68	76 83 20 37 90	57 16 00 11 66
14 90 84 45 11	75 73 88 05 90	52 27 41 14 86	22 98 12 22 08	07 52 74 95 80
68 05 51 18 00	33 96 02 75 19	07 60 62 93 55	59 33 82 43 90	49 37 38 44 59
20 46 78 73 90	97 51 40 14 02	04 02 33 31 08	39 54 16 49 36	47 95 93 13 30
64 19 58 97 79	15 06 15 93 20	01 90 10 75 06	40 78 78 89 62	02 67 74 17 33
05 26 93 70 60	22 35 85 15 13	92 03 51 59 77	59 56 78 06 83	52 91 05 70 74
07 97 10 88 23	09 98 42 99 64	61 71 62 99 15	06 51 29 16 93	58 05 77 09 51
68 71 86 85 85	54 87 66 47 54	73 32 08 11 12	44 95 92 63 16	29 56 24 29 48
26 99 61 65 53	58 37 78 80 70	42 10 50 67 42	32 17 55 85 74	94 44 67 16 94
14 65 52 68 75	87 59 36 22 41	26 78 63 06 55	13 08 27 01 50	15 29 39 39 43
17 53 77 58 71	71 41 61 50 72	12 41 94 96 26	44 95 27 36 99	02 96 74 30 83
90 26 59 21 19	23 52 23 33 12	96 93 02 18 39	07 02 18 36 07	25 99 32 70 23
41 23 52 55 99	31 04 49 69 96	10 47 48 45 88	13 41 43 89 20	97 17 14 49 17
60 20 50 81 69	31 99 73 68 68	35 81 33 03 76	24 30 12 48 60	18 99 10 72 34
91 25 38 05 90	94 58 28 41 36	45 37 59 03 09	90 35 57 29 12	82 62 54 65 60
34 50 57 74 37	98 80 33 00 91	09 77 93 19 82	74 94 80 04 04	45 07 31 66 49
85 22 04 39 43	73 81 53 94 79	33 62 46 86 28	08 31 54 46 31	53 94 13 38 47
09 79 13 77 48	73 82 97 22 21	05 03 27 24 83	72 89 44 05 60	35 80 39 94 88
88 75 80 18 14	22 95 75 42 49	39 32 82 22 49	02 48 07 70 37	16 04 61 67 87
90 96 23 70 00	39 00 03 06 90	55 85 78 38 36	94 37 30 69 32	90 89 00 76 33

Table XXXIII. Random Numbers (II)

53 74 23 99 67	61 32 28 69 84	94 62 67 86 24	98 33 41 19 95	47 53 53 38 09
63 38 06 86 54	99 00 65 26 94	02 82 90 23 07	79 62 67 80 60	75 91 12 81 19
35 30 58 21 46	06 72 17 10 94	25 21 31 75 96	49 28 24 00 49	55 65 79 78 07
63 43 36 82 69	65 51 18 37 88	61 38 44 12 45	32 92 85 88 65	54 34 81 85 35
98 25 37 55 26	01 91 82 81 46	74 71 12 94 97	24 02 71 37 07	03 92 18 66 75
02 63 21 17 69	71 50 80 89 56	38 15 70 11 48	43 40 45 86 98	00 83 26 91 03
64 55 22 21 82	48 22 28 06 00	61 54 13 43 91	82 78 12 23 29	06 66 24 12 27
85 07 26 13 89	01 10 07 82 04	59 63 69 36 03	69 11 15 83 80	13 29 54 19 28
58 54 16 24 15	51 54 44 82 00	62 61 65 04 69	38 18 65 18 97	85 72 13 49 21
34 85 27 84 87	61 48 64 56 26	90 18 48 13 26	37 70 15 42 57	65 65 80 39 07
03 92 18 27 46	57 99 16 96 56	30 33 72 85 22	84 64 38 56 98	99 01 30 98 64
62 95 30 27 59	37 75 41 66 48	86 97 80 61 45	23 53 04 01 63	45 76 08 64 27
08 45 93 15 22	60 21 75 46 91	98 77 27 85 42	28 88 61 08 84	69 62 03 42 73
07 08 55 18 40	45 44 75 13 90	24 94 96 61 02	57 55 66 83 15	73 42 37 11 61
01 85 89 95 66	51 10 19 34 88	15 84 97 19 75	12 76 39 43 78	64 63 91 08 25
72 84 71 14 35	19 11 58 49 26	50 11 17 17 76	86 31 57 20 18	95 60 78 46 75
88 78 28 16 84	13 52 53 94 53	75 45 69 30 96	73 89 65 70 31	99 17 43 48 76
45 17 75 65 57	28 40 19 72 12	25 12 74 75 67	60 40 60 81 19	24 62 01 61 16
96 76 28 12 54	22 01 11 94 25	71 96 16 16 88	68 64 36 74 45	19 59 50 88 92
43 31 67 72 30	24 02 94 08 63	38 32 36 66 02	69 36 38 25 39	48 03 45 15 22
50 44 66 44 21	66 06 58 05 62	68 15 54 35 02	42 35 48 96 32	14 52 41 52 48
22 66 22 15 86	26 63 75 41 99	58 42 36 72 24	58 37 52 18 51	03 37 18 39 11
96 24 40 14 51	23 22 30 88 57	95 67 47 29 83	94 69 40 06 07	18 16 36 78 86
31 73 91 61 19	60 20 72 93 48	98 57 07 23 69	65 95 39 69 58	56 80 30 19 44
78 60 73 99 84	43 89 94 36 45	56 69 47 07 41	90 22 91 07 12	78 35 34 08 72
84 37 90 61 56	70 10 23 98 05	85 11 34 76 60	76 48 45 34 60	01 64 18 39 96
36 67 10 08 23	98 93 35 08 86	99 29 76 29 81	33 34 91 58 93	63 14 52 32 52
07 28 59 07 48	89 64 58 89 75	83 85 62 27 89	30 14 78 56 27	86 63 59 80 02
10 15 83 87 60	79 24 31 66 56	21 48 24 06 93	91 98 94 05 49	01 47 59 38 00
55 19 68 97 65	03 73 52 16 56	00 53 55 90 27	38 42 29 38 87	22 13 88 83 34
53 81 29 13 39	35 01 20 71 34	62 33 74 82 14	53 73 19 09 03	56 54 29 56 93
51 86 32 68 92	33 98 74 66 99	40 14 71 94 58	45 94 19 38 81	14 44 99 81 07
35 91 70 29 13	80 03 54 07 27	96 94 78 32 66	50 95 52 74 33	13 80 55 62 54
37 71 67 95 13	20 02 44 95 94	64 85 04 05 72	01 32 90 76 14	53 89 74 60 41
93 66 13 83 27	92 79 64 64 72	28 54 96 53 84	48 14 52 98 94	56 07 93 89 30
02 96 08 45 65	13 05 00 41 84	93 07 54 72 59	21 45 57 09 77	19 48 56 27 44
49 83 43 48 35	82 88 33 69 96	72 36 04 19 76	47 45 15 18 60	82 11 08 95 97
84 60 71 62 46	40 80 81 30 37	34 39 23 05 38	25 15 35 71 30	88 12 57 21 77
18 17 30 88 71	44 91 14 88 47	89 23 30 63 15	56 34 20 47 89	99 82 93 24 98
79 69 10 61 78	71 32 76 95 62	87 00 22 58 40	92 54 01 75 25	43 11 71 99 31
75 93 36 57 83	56 20 14 82 11	74 21 97 90 65	96 42 68 63 86	74 54 13 26 94
38 30 92 29 03	06 28 81 39 38	62 25 06 84 63	61 29 08 93 67	04 32 92 08 09
51 29 50 10 34	31 57 75 95 80	51 97 02 74 77	76 15 48 49 44	18 55 63 77 09
21 31 38 86 24	37 79 81 53 74	73 24 16 10 33	52 83 90 94 76	70 47 14 54 36
29 01 23 87 88	58 02 39 37 67	42 10 14 20 92	16 55 23 42 45	54 96 09 11 06
95 33 95 22 00	18 74 72 00 18	38 79 58 69 32	81 76 80 26 92	82 80 84 25 39
90 84 60 79 80	24 36 59 87 38	82 07 53 89 35	96 35 23 79 18	05 98 90 07 35
46 40 62 98 82	54 97 20 56 95	15 74 80 08 32	16 46 70 50 80	67 72 16 42 79
20 31 89 03 43	38 46 82 68 72	32 14 82 99 70	80 60 47 18 97	63 49 30 21 30
71 59 73 05 50	08 22 23 71 77	91 01 93 20 49	82 96 59 26 94	66 39 67 98 60

TABLE XXXIII. RANDOM NUMBERS (III)

22 17 68 65 84	68 95 23 92 35	87 02 22 57 51	61 09 43 95 06	58 24 82 03 47
19 36 27 59 46	13 79 93 37 55	39 77 32 77 09	85 52 05 30 62	47 83 51 62 74
16 77 23 02 77	09 61 87 25 21	28 06 24 25 93	16 71 13 59 78	23 05 47 47 25
78 43 76 71 61	20 44 90 32 64	97 67 63 99 61	46 38 03 93 22	69 81 21 99 21
03 28 28 26 08	73 37 32 04 05	69 30 16 09 05	88 69 58 28 99	35 07 44 75 47
93 22 53 64 39	07 10 63 76 35	87 03 04 79 88	08 13 13 85 51	55 34 57 72 69
78 76 58 54 74	92 38 70 96 92	52 06 79 79 45	82 63 18 27 44	69 66 92 19 09
23 68 35 26 00	99 53 93 61 28	52 70 05 48 34	56 65 05 61 86	90 92 10 70 80
15 39 25 70 99	93 86 52 77 65	15 33 59 05 28	22 87 26 07 47	86 96 98 29 06
58 71 96 30 24	18 46 23 34 27	85 13 99 24 44	49 18 09 79 49	74 16 32 23 02
57 35 27 33 72	24 53 63 94 09	41 10 76 47 91	44 04 95 49 66	39 60 04 59 81
48 50 86 54 48	22 06 34 72 52	82 21 15 65 20	33 29 94 71 11	15 91 29 12 03
61 96 48 95 03	07 16 39 33 66	98 56 10 56 79	77 21 30 27 12	90 49 22 23 62
36 93 89 41 26	29 70 83 63 51	99 74 20 52 36	87 09 41 15 09	98 60 16 03 03
18 87 00 42 31	57 90 12 02 07	23 47 37 17 31	54 08 01 88 63	39 41 88 92 10
88 56 53 27 59	33 35 72 67 47	77 34 55 45 70	08 18 27 38 90	16 95 86 70 75
09 72 95 84 29	49 41 31 06 70	42 38 06 45 18	64 84 73 31 65	52 53 37 97 15
12 96 88 17 31	65 19 69 02 83	60 75 86 90 68	24 64 19 35 51	56 61 87 39 12
85 94 57 24 16	92 09 84 38 76	22 00 27 69 85	29 81 94 78 70	21 94 47 90 12
38 64 43 59 98	98 77 87 68 07	91 51 67 62 44	40 98 05 93 78	23 32 65 41 18
53 44 09 42 72	00 41 86 79 79	68 47 22 00 20	35 55 31 51 51	00 83 63 22 55
40 76 66 26 84	57 99 99 90 37	36 63 32 08 58	37 40 13 68 97	87 64 81 07 83
02 17 79 18 05	12 59 52 57 02	22 07 90 47 03	28 14 11 30 79	20 69 22 40 98
95 17 82 06 53	31 51 10 96 46	92 06 88 07 77	56 11 50 81 69	40 23 72 51 39
35 76 22 42 92	96 11 83 44 80	34 68 35 48 77	33 42 40 90 60	73 96 53 97 86
26 29 13 56 41	85 47 04 66 08	34 72 57 59 13	82 43 80 46 15	38 26 61 70 04
77 80 20 75 82	72 82 32 99 90	63 95 73 76 63	89 73 44 99 05	48 67 26 43 18
46 40 66 44 52	91 36 74 43 53	30 82 13 54 90	78 45 63 98 35	55 03 36 67 68
37 56 08 18 09	77 53 84 46 47	31 91 18 95 58	24 16 74 11 53	44 10 13 85 57
61 65 61 68 66	37 27 47 39 19	84 83 70 07 48	53 21 40 06 71	95 06 79 88 54
93 43 69 64 07	34 18 04 52 35	56 27 09 24 86	61 85 53 83 45	19 90 70 99 00
21 96 60 12 99	11 20 99 45 18	48 13 93 55 34	18 37 79 49 90	65 97 38 20 46
95 20 47 97 97	27 37 83 28 71	00 06 41 41 74	45 89 09 39 84	51 67 11 52 49
97 86 21 78 73	10 65 81 92 59	58 76 17 14 97	04 76 62 16 17	17 95 70 45 80
69 92 06 34 13	59 71 74 17 32	27 55 10 24 19	23 71 82 13 74	63 52 52 01 41
04 31 17 21 56	33 73 99 19 87	26 72 39 27 67	53 77 57 68 93	60 61 97 22 61
61 06 98 03 91	87 14 77 43 96	43 00 65 98 50	45 60 33 01 07	98 99 46 50 47
85 93 85 86 88	72 87 08 62 40	16 06 10 89 20	23 21 34 74 97	76 38 03 29 63
21 74 32 47 45	73 96 07 94 52	09 65 90 77 47	25 76 16 19 33	53 05 70 53 30
15 69 53 82 80	79 96 23 53 10	65 39 07 16 29	45 33 02 43 70	02 87 40 41 45
02 89 08 04 49	20 21 14 68 86	87 63 93 95 17	11 29 01 95 80	35 14 97 35 33
87 18 15 89 79	85 43 01 72 73	08 61 74 51 69	89 74 39 82 15	94 51 33 41 67
98 83 71 94 22	59 97 50 99 52	08 52 85 08 40	87 80 61 65 31	91 51 80 32 44
10 08 58 21 66	72 68 49 29 31	89 85 84 46 06	59 73 19 85 23	65 09 29 75 63
47 90 56 10 08	88 02 84 27 83	42 29 72 23 19	66 56 45 65 79	20 71 53 20 25
22 85 61 68 90	49 64 92 85 44	16 40 12 89 88	50 14 49 81 06	01 82 77 45 12
67 80 43 79 33	12 83 11 41 16	25 58 19 68 70	77 02 54 00 52	53 43 37 15 26
27 62 50 96 72	79 44 61 40 15	14 53 40 65 39	27 31 58 50 28	11 39 03 34 25
33 78 80 87 15	38 30 06 38 21	14 47 47 07 26	54 96 87 53 32	40 36 40 96 76
13 13 92 66 99	47 24 49 57 74	32 25 43 62 17	10 97 11 69 84	99 63 22 32 98

TABLE XXXIII. RANDOM NUMBERS (IV)

10 27 53 96 23	71 50 54 36 23	54 31 04 82 98	04 14 12 15 09	26 78 25 47 47
28 41 50 61 88	64 85 27 20 18	83 36 36 05 56	39 71 65 09 62	94 76 62 11 89
34 21 42 57 02	59 19 18 97 48	80 30 03 30 98	05 24 67 70 07	84 97 50 87 46
61 81 77 23 23	82 82 11 54 08	53 28 70 58 96	44 07 39 55 43	42 34 43 39 28
61 15 18 13 54	16 86 20 26 88	90 74 80 55 09	14 53 90 51 17	52 01 63 01 59
91 76 21 64 64	44 91 13 32 97	75 31 62 66 54	84 80 32 75 77	56 08 25 70 29
00 97 79 08 06	37 30 28 59 85	53 56 68 53 40	01 74 39 59 73	30 19 99 85 48
36 46 18 34 94	75 20 80 27 77	78 91 69 16 00	08 43 18 73 68	67 69 61 34 25
88 98 99 60 50	65 95 79 42 94	93 62 40 89 96	43 56 47 71 66	46 76 29 67 02
04 37 59 87 21	05 02 03 24 17	47 97 81 56 51	92 34 86 01 82	55 51 33 12 91
63 62 06 34 41	94 21 78 55 09	72 76 45 16 94	29 95 81 83 83	79 88 01 97 30
78 47 23 53 90	34 41 92 45 71	09 23 70 70 07	12 38 92 79 43	14 85 11 47 23
87 68 62 15 43	53 14 36 59 25	54 47 33 70 15	59 24 48 40 35	50 03 42 99 36
47 60 92 10 77	88 59 53 11 52	66 25 69 07 04	48 68 64 71 06	61 65 70 22 12
56 88 87 59 41	65 28 04 67 53	95 79 88 37 31	50 41 06 94 76	81 83 17 16 33
02 57 45 86 67	73 43 07 34 48	44 26 87 93 29	77 09 61 67 84	06 69 44 77 75
31 54 14 13 17	48 62 11 90 60	68 12 93 64 28	46 24 79 16 76	14 60 25 51 01
28 50 16 43 36	28 97 85 58 99	67 22 52 76 23	24 70 36 54 54	59 28 61 71 96
63 29 62 66 50	02 63 45 52 38	67 63 47 54 75	83 24 78 43 20	92 63 13 47 48
45 65 58 26 51	76 96 59 38 72	86 57 45 71 46	44 67 76 14 55	44 88 01 62 12
39 65 36 63 70	77 45 85 50 51	74 13 39 35 22	30 53 36 02 95	49 34 88 73 61
73 71 98 16 04	29 18 94 51 23	76 51 94 84 86	79 93 96 38 63	08 58 25 58 94
72 20 56 20 11	72 65 71 08 86	79 57 95 13 91	97 48 72 66 48	09 71 17 24 89
75 17 26 99 76	89 37 20 70 01	77 31 61 95 46	26 97 05 73 51	53 33 18 72 87
37 48 60 82 29	81 30 15 39 14	48 38 75 93 29	06 87 37 78 48	45 56 00 84 47
68 08 02 80 72	83 71 46 30 49	89 17 95 88 29	02 39 56 03 46	97 74 06 56 17
14 23 98 61 67	70 52 85 01 50	01 84 02 78 43	10 62 98 19 41	18 83 99 47 99
49 08 96 21 44	25 27 99 41 28	07 41 08 34 66	19 42 74 39 91	41 96 53 78 72
78 37 06 08 43	63 61 62 42 29	39 68 95 10 96	09 24 23 00 62	56 12 80 73 16
37 21 34 17 68	68 96 83 23 56	32 84 60 15 31	44 73 67 34 77	91 15 79 74 58
14 29 09 34 04	87 83 07 55 07	76 58 30 83 64	87 29 25 58 84	86 50 60 00 25
58 43 28 06 36	49 52 83 51 14	47 56 91 29 34	05 87 31 06 95	12 45 57 09 09
10 43 67 29 70	80 62 80 03 42	10 80 21 38 84	90 56 35 03 09	43 12 74 49 14
44 38 88 39 54	86 97 37 44 22	00 95 01 31 76	17 16 29 56 63	38 78 94 49 81
90 69 59 19 51	85 39 52 85 13	07 28 37 07 61	11 16 36 27 03	78 86 72 04 95
41 47 10 25 62	97 05 31 03 61	20 26 36 31 62	68 69 86 95 44	84 95 48 46 45
91 94 14 63 19	75 89 11 47 11	31 56 34 19 09	79 57 92 36 59	14 93 87 81 40
80 06 54 18 66	09 18 94 06 19	98 40 07 17 81	22 45 44 84 11	24 62 20 42 31
67 72 77 63 48	84 08 31 55 58	24 33 45 77 58	80 45 67 93 82	75 70 16 08 24
59 40 24 13 27	79 26 88 86 30	01 31 60 10 39	53 58 47 70 93	85 81 56 39 38
05 90 35 89 95	01 61 16 96 94	50 78 13 69 36	37 68 53 37 31	71 26 35 03 71
44 43 80 69 98	46 68 05 14 82	90 78 50 05 62	77 79 13 57 44	59 60 10 39 66
61 81 31 96 82	00 57 25 60 59	46 72 60 18 77	55 66 12 62 11	08 99 55 64 57
42 88 07 10 05	24 98 65 63 21	47 21 61 88 32	27 80 30 21 60	10 92 35 36 12
77 94 30 05 39	28 10 99 00 27	12 73 73 99 12	49 99 57 94 82	96 88 57 17 91
78 83 19 76 16	94 11 68 84 26	23 54 20 86 85	23 86 66 99 07	36 37 34 92 09
87 76 59 61 81	43 63 64 61 61	65 76 36 95 90	18 48 27 45 68	27 23 65 30 72
91 43 05 96 47	55 78 99 95 24	37 55 85 78 78	01 48 41 19 10	35 19 54 07 73
84 97 77 72 73	09 62 06 65 72	87 12 49 03 60	41 15 20 76 27	50 47 02 29 16
87 41 60 76 83	44 88 96 07 80	83 05 83 38 96	73 70 66 81 90	30 56 10 48 59

TABLE XXXIII. RANDOM NUMBERS (V)

28 89 65 87 08	13 50 63 04 23	25 47 57 91 13	52 62 24 19 94	91 67 48 57 10
30 29 43 65 42	78 66 28 55 80	47 46 41 90 08	55 98 78 10 70	49 92 05 12 07
95 74 62 60 53	51 57 32 22 27	12 72 72 27 77	44 67 32 23 13	67 95 07 76 30
01 85 54 96 72	66 86 65 64 60	56 59 75 36 75	46 44 33 63 71	54 50 06 44 75
10 91 46 96 86	19 83 52 47 53	65 00 51 93 51	30 80 05 19 29	56 23 27 19 03
05 33 18 08 51	51 78 57 26 17	34 87 96 23 95	89 99 93 39 79	11 28 94 15 52
04 43 13 37 00	79 68 96 26 60	70 39 83 66 56	62 03 55 86 57	77 55 33 62 02
05 85 40 25 24	73 52 93 70 50	48 21 47 74 63	17 27 27 51 26	35 96 29 00 45
84 90 90 65 77	63 99 25 69 02	09 04 03 35 78	19 79 95 07 21	02 84 48 51 97
28 55 53 09 48	86 28 30 02 35	71 30 32 06 47	93 74 21 86 33	49 90 21 69 74
89 83 40 69 80	97 96 47 59 97	56 33 24 87 36	17 18 16 90 46	75 27 28 52 13
73 20 96 05 68	93 41 69 96 07	97 50 81 79 59	42 37 13 81 83	92 42 85 04 31
10 89 07 76 21	40 24 74 36 42	40 33 04 46 24	35 63 02 31 61	34 59 43 36 96
91 50 27 78 37	06 06 16 25 98	17 78 80 36 85	26 41 77 63 37	71 63 94 94 33
03 45 44 66 88	97 81 26 03 89	39 46 67 21 17	98 10 39 33 15	61 63 00 25 92
89 41 58 91 63	65 99 59 97 84	90 14 79 61 55	56 16 88 87 60	32 15 99 67 43
13 43 00 97 26	16 91 21 32 41	60 22 66 72 17	31 85 33 69 07	68 49 20 43 29
71 71 00 51 72	62 03 89 26 32	35 27 99 18 25	78 12 03 09 70	50 93 19 35 56
19 28 15 00 41	92 27 73 40 38	37 11 05 75 16	98 81 99 37 29	92 20 32 39 67
56 38 30 92 30	45 51 94 69 04	00 84 14 36 37	95 66 39 01 09	21 68 40 95 79
39 27 52 89 11	00 81 06 28 48	12 08 05 75 26	03 35 63 05 77	13 81 20 67 58
73 13 28 58 01	05 06 42 24 07	60 60 29 99 93	72 93 78 04 36	25 76 01 54 03
81 60 84 51 57	12 68 46 55 89	60 09 71 87 89	70 81 10 95 91	83 79 68 20 66
05 62 98 07 85	07 79 26 69 61	67 85 72 37 41	85 79 76 48 23	61 58 87 08 05
62 97 16 29 18	52 16 16 23 56	62 95 80 97 63	32 25 34 03 36	48 84 60 37 65
31 13 63 21 08	16 01 92 58 21	48 79 74 73 72	08 64 80 91 38	07 28 66 61 59
97 38 35 34 19	89 84 05 34 47	88 09 31 54 88	97 96 86 01 69	46 13 95 65 96
32 11 78 33 82	51 99 98 44 39	12 75 10 60 36	80 66 39 94 97	42 36 31 16 59
81 99 13 37 05	08 12 60 39 23	61 73 84 89 18	26 02 04 37 95	96 18 69 06 30
45 74 00 03 05	69 99 47 26 52	48 06 30 00 18	03 30 28 55 59	66 10 71 44 05
11 84 13 69 01	88 91 28 79 50	71 42 14 96 55	98 59 96 01 36	88 77 90 45 59
14 66 12 87 22	59 45 27 08 51	85 64 23 85 41	64 72 08 59 44	67 98 36 65 56
40 25 67 87 82	84 27 17 30 37	48 69 49 02 58	98 02 50 58 11	95 39 06 35 63
44 48 97 49 43	65 45 53 41 07	14 83 46 74 11	76 66 63 60 08	90 54 33 65 84
41 94 54 06 57	48 28 01 83 84	09 11 21 91 73	97 28 44 74 06	22 30 95 69 72
07 12 15 58 84	93 18 31 83 45	54 52 62 29 91	53 58 54 66 05	47 19 63 92 75
64 27 90 43 52	18 26 32 96 83	50 58 45 27 57	14 96 39 64 85	73 87 96 76 23
80 71 86 41 03	45 62 63 40 88	35 69 34 10 94	32 22 52 04 74	69 63 21 83 41
27 06 08 09 92	26 22 59 28 27	38 58 22 14 79	24 32 12 38 42	33 56 90 92 57
54 68 97 20 54	33 26 74 03 30	74 22 19 13 48	30 28 01 92 49	58 61 52 27 03
02 92 65 68 99	05 53 15 26 70	04 69 22 64 07	04 73 25 74 82	78 35 22 21 88
83 52 57 78 62	98 61 70 48 22	68 50 64 55 75	42 70 32 09 60	58 70 61 43 97
82 82 76 31 33	85 13 41 38 10	16 47 61 43 77	83 27 19 70 41	34 78 77 60 25
38 61 34 09 49	04 41 66 09 76	20 50 73 40 95	24 77 95 73 20	47 42 80 61 03
01 01 11 88 38	03 10 16 82 24	39 58 20 12 39	82 77 02 18 88	33 11 49 15 16
21 66 14 38 28	54 08 18 07 04	92 17 63 36 75	33 14 11 11 78	97 30 53 62 38
32 29 30 69 59	68 50 33 31 47	15 64 88 75 27	04 51 41 61 96	86 62 93 66 71
04 59 21 65 47	39 90 89 86 77	46 86 86 88 86	50 09 13 24 91	54 80 67 78 66
38 64 50 07 36	56 50 45 94 25	48 28 48 30 51	60 73 73 03 87	68 47 37 10 84
48 33 50 83 53	59 77 64 59 90	58 92 62 50 18	93 09 45 89 06	13 26 98 86 29

TABLE XXXIII. RANDOM NUMBERS (VI)

25 19 64 82 84	62 74 29 92 24	61 03 91 22 48	64 94 63 15 07	66 85 12 00 27
23 02 41 46 04	44 31 52 43 07	44 06 03 09 34	19 83 94 62 94	48 28 01 51 92
55 85 66 96 28	28 30 62 58 83	65 68 62 42 45	13 08 60 46 28	95 68 45 52 43
68 45 19 69 59	35 14 82 56 80	22 06 52 26 39	59 78 98 76 14	36 09 03 01 86
69 31 46 29 85	18 88 26 95 54	01 02 14 03 05	48 00 26 43 85	33 93 81 45 95
37 31 61 28 98	94 61 47 03 10	67 80 84 41 26	88 84 59 69 14	77 32 82 81 89
66 42 19 24 94	13 13 38 69 96	76 69 76 24 13	43 83 10 13 24	18 32 84 85 04
33 65 78 12 35	91 59 11 38 44	23 31 48 75 74	05 30 08 46 32	90 04 93 56 16
76 32 06 19 35	22 95 30 19 29	57 74 43 20 90	20 25 36 70 69	38 32 11 01 01
43 33 42 02 59	20 39 84 95 61	58 22 04 02 99	99 78 78 83 82	43 67 16 38 95
28 31 93 43 94	87 73 19 38 47	54 36 90 98 10	83 43 32 26 26	22 00 90 59 22
97 19 21 63 34	69 33 17 03 02	11 15 50 46 08	42 69 60 17 42	14 68 61 14 48
82 80 37 14 20	56 39 59 89 63	33 90 38 44 50	78 22 87 10 88	06 58 87 39 67
03 68 03 13 60	64 13 09 37 11	86 02 57 41 99	31 66 60 65 64	03 03 02 58 97
65 16 58 11 01	98 78 80 63 23	07 37 66 20 56	20 96 06 79 80	33 39 40 49 42
24 65 58 57 04	18 62 85 28 24	26 45 17 82 76	39 65 01 73 91	50 37 49 38 73
02 72 64 07 75	85 66 48 38 73	75 10 96 59 31	48 78 58 08 88	72 08 54 57 17
79 16 78 63 99	43 61 00 66 42	76 26 71 14 33	33 86 76 71 66	37 85 05 56 07
04 75 14 93 39	68 52 16 83 34	64 09 44 62 58	48 32 72 26 95	32 67 35 49 71
40 64 64 57 60	97 00 12 91 33	22 14 73 01 11	83 97 68 95 65	67 77 80 98 87
06 27 07 34 26	01 52 48 69 57	19 17 53 55 96	02 41 03 89 33	86 85 73 02 32
62 40 03 87 10	96 88 22 46 94	35 56 60 94 20	60 73 04 84 98	96 45 18 47 07
00 98 48 18 97	91 51 63 27 95	74 25 84 03 07	88 29 04 79 84	03 71 13 78 26
50 64 19 18 91	98 55 83 46 09	49 66 41 12 45	41 49 36 83 43	53 75 35 13 39
38 54 52 25 78	01 98 00 89 85	86 12 22 89 25	10 10 71 19 45	88 84 77 00 07
46 86 80 97 78	65 12 64 64 70	58 41 05 49 08	68 68 88 54 00	81 61 61 80 41
90 72 92 93 10	09 12 81 93 63	69 30 02 04 26	92 36 48 69 45	91 99 08 07 65
66 21 41 77 60	99 35 72 61 22	52 40 74 67 29	97 50 71 39 79	57 82 14 88 06
87 05 46 52 76	89 96 34 22 37	27 11 57 04 19	57 93 08 35 69	07 51 19 92 66
46 90 61 03 06	89 85 33 22 80	34 89 12 29 37	44 71 38 40 37	15 49 55 51 08
11 88 53 06 09	81 83 33 98 29	91 27 59 43 09	70 72 51 49 73	35 97 25 83 41
11 05 92 06 97	68 82 34 08 83	25 40 58 40 64	56 42 78 54 06	60 96 96 12 82
33 94 24 20 28	62 42 07 12 63	34 39 02 92 31	80 61 68 44 19	09 92 14 73 49
24 89 74 75 61	61 02 73 36 85	67 28 50 49 85	37 79 95 02 66	73 19 76 28 13
15 19 74 67 23	61 38 93 73 68	76 23 15 58 20	35 36 82 82 59	01 33 48 17 66
05 64 12 70 88	80 58 35 06 88	73 48 27 39 43	43 40 13 35 45	55 10 54 38 50
57 49 36 44 06	74 93 55 39 26	27 70 98 76 68	78 36 26 24 06	43 24 56 40 80
77 82 96 96 97	60 42 17 18 48	16 34 92 19 52	98 84 48 42 92	83 19 06 77 78
24 10 70 06 51	59 62 37 95 42	53 67 14 95 29	84 65 43 07 30	77 54 00 15 42
50 00 07 78 23	49 54 36 85 14	18 50 54 18 82	23 79 80 71 37	60 62 95 40 30
44 37 76 21 96	37 03 08 98 64	90 85 59 43 64	17 79 96 52 35	21 05 22 59 30
90 57 55 17 47	53 26 79 20 38	69 90 58 64 03	33 48 32 91 54	68 44 90 24 25
50 74 64 67 42	95 28 12 73 23	32 54 98 64 94	82 17 18 17 14	55 10 61 64 29
44 04 70 22 02	84 31 64 64 08	52 55 04 24 29	91 95 43 81 14	66 13 18 47 44
32 74 61 64 73	21 46 51 44 77	72 48 92 00 05	83 59 89 65 06	53 76 70 58 78
75 73 51 70 49	12 53 67 51 54	38 10 11 67 73	22 32 61 43 75	31 61 22 21 11
76 18 36 16 34	16 28 25 82 98	64 26 70 54 87	49 48 55 11 39	94 25 20 80 85
00 17 37 71 81	64 21 91 15 82	81 04 14 52 11	39 07 30 60 77	39 18 27 85 68
54 95 57 55 04	12 77 40 70 14	79 86 61 57 50	52 49 41 73 46	05 63 34 92 33
69 99 95 54 63	44 37 33 53 17	38 06 58 37 93	47 10 62 31 28	63 59 40 40 32

Table XXXIV. Constants, Weights and Measures, Etc.

Mathematical Constants

$\pi = 3{\cdot}14159\ 26535\ 89793\ 23846$	$e = 2{\cdot}71828\ 18284\ 59045\ 23536$	$\gamma = 0{\cdot}57721\ 56649\ 01532\ 86061$
$\log_{10}\pi = 0{\cdot}49714\ 98726\ 94133\ 85435$	$\log_{10}e = 0{\cdot}43429\ 44819\ 03251\ 82765$	$e^{\gamma} = 1{\cdot}78107\ 24179\ 90197\ 98522$
$1/\sqrt{2\pi} = 0{\cdot}39894\ 22804\ 01432\ 67794$	$\log_{e}10 = 2{\cdot}30258\ 50929\ 94045\ 68402$	$e^{-\gamma} = 0{\cdot}56145\ 94835\ 66885\ 16983$

Physical Constants

Velocity of light $= 3{\cdot}00 \times 10^{10}$ cm./sec. = 186,000 miles/sec. The first number is the ratio of electromagnetic to electrostatic unit of charge.

Ionic (electronic) charge (e) $= 4{\cdot}77 \times 10^{-10}$ E.S.U. $= 1{\cdot}59 \times 10^{-20}$ E.M.U. Mass of electron (m_0) $= 9{\cdot}00 \times 10^{-28}$ g.

Planck's constant (h) $= 6{\cdot}55 \times 10^{-27}$ erg secs.

Energy of gas molecule per 1° C. absolute $= 2{\cdot}06 \times 10^{-16}$ ergs.

Number of molecules in 1 gm. molecule of gas $= 6{\cdot}06 \times 10^{23}$.

Electrical charge of 1 gm. equivalent in electrolysis (Faraday) $= 9{\cdot}65 \times 10^{3}$ E.M.U.

Gas constant (R) $= 8{\cdot}315 \times 10^{7}$ c.g.s. ($pv = RT \times$ mass in gram molecules).

Constant of gravitation (G) $= 6{\cdot}67 \times 10^{-8}$ c.g.s. ($f = Gmm'/r^2$).

Acceleration due to gravity (g). Teddington (N. 51° 28·1′): 981·181 cm./sec.^{2} = 32·191 ft./sec.^{2}; Washington (38° 56·5′): 980·082; whence at lat. λ, height H m.: $980{\cdot}618 - 2{\cdot}586 \cos 2\lambda + {\cdot}003 \cos 4\lambda$, $-{\cdot}0003086\ H$.

The earth. Equatorial radius = 3963·4 miles, polar radius = 3950·0 miles. 1° of Latitude = 68·70 mls. (equator), 69·41 mls. (pole). Mean density = 345 lb./ft.^{3} = 5·517 g./cm.^{3}

Distance of sun $= 93 \times 10^{6}$ miles. Distance of moon = 239,000 miles.

Solar radiation (Washington, D.C.) = ·035 cal./cm.^{2}/sec.

Velocity of sound (6° C.) $= 3{\cdot}35 \times 10^{4}$ cm./sec. = 1100 ft./sec. Audible sound: Lower limit, 30 vibr./sec.; upper limit, 24,000-41,000 vibr./sec. Middle C of piano, 261 vibr./sec. Wave-lengths of visible light: 4 to 7×10^{-5} cm.

Temperature Equivalents												
Centigrade	−40	−30	−20	−10	**0**	10	20	30	40	50	**100**
Fahrenheit	−40	−22	−4	14	**32**	50	68	86	104	122	**212**
Réaumur	−32	−24	−16	−8	**0**	8	16	24	32	49	**80**

1° F. $= \frac{5}{9}$° C. 1° R. $= \frac{5}{4}$° C.

Absolute zero of temperature = −273·1° C.

Mechanical and Electrical Units

Force. g dynes = 1 gm. weight, g poundals = 1 lb. weight, where g is acceleration due to gravity measured in appropriate units (cm./sec.^{2} or ft./sec.^{2}). 1 poundal $= 1{\cdot}3825 \times 10^{4}$ dynes.

Pressure. 1 atmosphere = 760 mm. (29·921 ins.) of mercury (0° C., Lat. 45°) = 33·899 ft. of water (4° C.) = 14·696 lb./in.^{2} $= 1{\cdot}01325 \times 10^{6}$ dynes/cm.^{2} 1 bar $= 10^{6}$ dynes/cm.^{2} = ·98692 atmos. 1 lb./in.^{2} = 70·307 (1/·014223) gm./cm.^{2}

Work, Power, etc. 1 erg = 1 dyne cm. 1 joule $= 10^{7}$ ergs = 1 watt second. 1 foot pound $= 1{\cdot}356 \times 10^{7}$ ergs. 1 horse power (H.P.) = 550 ft. lb. per sec. = 746 watts (continental H.P. = 736 watts). 1 calorie = heat to raise 1 gm. of water 1° C. at a specified temperature. 1 15° C. calorie = 1 mean calorie (0° C.-100° C.) $= 4{\cdot}185 \times 10^{7}$ ergs. 1 British Thermal Unit (1 lb. water × 1° F.) = 252·0 calories $= 2{\cdot}930 \times 10^{-4}$ kilowatt hours $= 3{\cdot}929 \times 10^{-4}$ horse power hours $= 10^{-5}$ gas therms. 1 Board of Trade Unit = 1 kilowatt hour.

Electrical. Relations between engineering units and electromagnetic units. Charge: coulomb $= 10^{-1}$ E.M.U. Current: ampere $= 10^{-1}$ E.M.U. Potential: volt $= 10^{8}$ E.M.U. Resistance: ohm $= 10^{9}$ E.M.U. Capacity: farad $= 10^{-9}$ E.M.U. Inductance: henry $= 10^{9}$ E.M.U. Flux: maxwell $= 10^{0}$ E.M.U. Magnetic force: gauss $= 10^{0}$ E.M.U.

Supposed Duration of Geologic Periods (E. W. Barnes)

		Duration Million Years.			Duration Million Years.
Tertiary or Cainozoic era	Pleistocene, Pliocene, Miocene, Oligocene, Eocene	60	Newer Palæozoic era	Permian, Carboniferous, Devonian	175
Age of Mammals					
Secondary or Mesozoic era	Cretaceous, Jurassic, Triassic	85	Older Palæozoic era	Silurian, Ordovician, Cambrian	250
Age of reptiles			Archæozoic era		300
			Eozoic era		500

Possible age of planet 2000-4000 million years.

TABLE XXXIV. CONSTANTS, WEIGHTS AND MEASURES, ETC.—*continued*

The Elements and their Atomic Weights

H	Hydrogen	1·0080	Cr	Chromium	52·01	Ag	Silver	107·880	Yb	Ytterbium	173·04
He	Helium	4·003	Mn	Manganese	54·93	Cd	Cadmium	112·41	Lu	Lutecium	174·99
Li	Lithium	6·940	Fe	Iron	55·85	In	Indium	114·76	Hf	Hafnium	178·6
Be	Beryllium	9·02	Co	Cobalt	58·94	Sn	Tin	118·70	Ta	Tantalum	180·88
B	Boron	10·82	Ni	Nickel	58·69	Sb	Antimony	121·76	W	Tungsten	183·92
C	Carbon	12·010	Cu	Copper	63·57	Te	Tellurium	127·61	Re	Rhenium	186·31
N	Nitrogen	14·008	Zn	Zinc	65·38	I	Iodine	126·92	Os	Osmium	190·2
O	Oxygen	16·000	Ga	Gallium	69·72	Xe	Xenon	131·3	Ir	Iridium	193·1
F	Fluorine	19·00	Ge	Germanium	72·60	Cs	Cesium	132·91	Pt	Platinum	195·23
Ne	Neon	20·183	As	Arsenic	74·91	Ba	Barium	137·36	Au	Gold	197·2
Na	Sodium	22·997	Se	Selenium	78·96	La	Lanthanum	138·92	Hg	Mercury	200·61
Mg	Magnesium	24·32	Br	Bromine	79·916	Ce	Cerium	140·13	Tl	Thallium	204·39
Al	Aluminium	26·97	Kr	Krypton	83·7	Pr	Praseodymium	140·92	Pb	Lead	207·21
Si	Silicon	28·06	Rb	Rubidium	85·48	Nd	Neodymium	144·27	Bi	Bismuth	209·00
P	Phosphorus	30·98	Sr	Strontium	87·63	—	—	—	Po	Polonium	(210)
S	Sulphur	32·06	Yt	Yttrium	88·92	Sa	Samarium	150·43	—	—	—
Cl	Chlorine	35·457	Zr	Zirconium	91·22	Eu	Europium	152·0	Rn	Radon	222
A	Argon	39·944	Cb	Columbium	92·91	Gd	Gadolinium	156·9	—	—	—
K	Potassium	39·096	Mo	Molybdenum	95·95	Tb	Terbium	159·2	Ra	Radium	226·05
Ca	Calcium	40·08	Ma	Masurium	?	Dy	Dysprosium	162·46	Ac	Actinium	?
Sc	Scandium	45·10	Ru	Ruthenium	101·7	Ho	Holmium	164·94	Th	Thorium	232·12
Ti	Titanium	47·90	Rh	Rhodium	102·91	Er	Erbium	167·2	Pa	Protoactinium	231
V	Vanadium	50·95	Pd	Palladium	106·7	Tu	Thulium	169·4	U	Uranium	238·07

Densities

Aluminium	2·70	Iron	7·86	Platinum	21·50	Coal	1·2-1·8	White pine	·4 - ·5	Glass	2·4 -4·5
Copper	8·93	Lead	11·37	Silver	10·5	Granite	2·5-3	Beech, oak	·7 - ·9	Sea water	1·01-1·05
Diamond	3·52	Magnesium	1·74	Tin	7·29	Quartz	2·66	Ebony	1·1 -1·3	Loose snow	·12 approx.
Gold	19·32	Mercury	13·56	Zinc	7·1	Slate	2·5-2·7	Cork	·22 - ·26	Soil	1·6 approx.

Unit density = 1 g. per cu. cm. = ·036128 lb. per cu. in.

Equation of Time

Jan.	1	+ 3m.	11s.	Mar.	1	+12m.	34s.	May	1	−2m.	57s.	July	1	+3m.	32s.	Sept. 16	− 5m. 6s.	Nov. 16	−15m. 10s.
,,	16	+ 9	33	,,	16	+ 8	51	,,	14	−3	49*	,,	26	+6	18*	Oct. 1	−10 16	Dec. 1	−10 56
Feb.	1	+13	37	Apr.	1	+ 4	1	June	1	−2	27	Aug.	16	+4	11	,, 16	−14 20	,, 12	− 6 15
,,	12	+14	25*	,,	16	0	0	,,	15	0	0	Sept.	1	0	0	Nov. 3	−16 21*	,, 25	0 0

When positive add the equation of time to apparent solar time to obtain mean solar time. The values given are mean values. Maxima and minima are marked with asterisks. 1 sidereal day = 23 h. 56 m. 4·09 s.

Saturated Vapour Pressure of Water (Millibars)

°C.	−50	−40	−30	−20	−10	−5	−2
Ice	·039	·129	·381	1·035	2·600	4·017	5·173
Water	—	—	—	1·270	2·865	4·217	5·274

°C.	0	1	2	3	4	5	6	7	8	9
0	6·105	6·567	7·058	7·579	8·134	8·723	9·350	10·02	10·73	11·48
10	12·28	13·12	14·02	14·97	15·98	17·05	18·18	19·37	20·63	21·97
20	23·38	24·86	26·43	28·09	29·83	31·67	33·61	35·65	37·80	40·05
30	42·43	44·92	47·55	50·31	53·19	56·23	59·41	62·75	66·25	69·92

40	73·76
50	123·3
60	199·2
70	311·6
80	473·4
90	701·0
100	1013·2
200	15544
300	85903

Critical temperature: 374°. Critical pressure: 220·6 bars. Interpolate logarithmically.

Composition of Fertilizers (Commercial Grades on the English Market)

	%N		%P_2O_5		%K_2O
Sulphate of ammonia	20·6	Superphosphate	16-18 water soluble	Sulphate of potash	48
Nitrate of soda	16	Basic slag	10-18	Muriate of potash	60
Nitrochalk	15½	Rock phosphate	26-28 total	Potash manure salt	20, 30
Calcium cyanamide	20·6	Triple superphosphate	48	Kainit	14

Many of these fertilizers are mixtures of two or more constituents in naturally occurring or conventional proportions. 1945 values are given.

English Weights and Measures

Length

1 foot = 12 ins. 1 yard = 3 ft.
1 mile = 1760 yds. = 5280 ft.
(60 m.p.h. = 88 ft. per sec.)

---

1 surveyor's chain = 22 yds.
(1 engineer's chain = 100 ft.)
1 link = $\frac{1}{100}$ chain = 7·92 ins.
(1 rod, pole, or perch = $\frac{1}{4}$ chain.)
1 furlong = 10 chains = $\frac{1}{8}$ mile.

---

1 fathom = 6 ft. 1 cable = 100 fath.
1 nautical mile = 6080 ft.
= 1 minute of arc of equator.
(1 knot = 1 nautical mile per hour.)

Archaic (approximate):
1 hand = 4 ins. 1 cubit = 18 ins.
1 ell = 45 ins. 1 league = 3 miles.

Area

1 acre = 10 sq. chains = 4840 sq. yds.
1 sq. mile = 640 acres.
(1 rod = $\frac{1}{160}$ ac., 1 rood = $\frac{1}{4}$ ac.)

Weight

Avoirdupois (General System)

1 ounce (oz.) = 16 drams (dm.).
1 pound (lb.) = 16 oz.
1 stone = 14 lb. 1 quarter = 28 lb.
1 hundredweight (cwt.) = 112 lb.
= 4 quarters = 8 stones.
1 ton = 20 cwt. = 2240 lb.

Apothecary (Drugs)	*Troy (Precious Metals)*
1 drachm = 60 grs.	1 pennyweight
(1 scruple = 20 grs.)	(dwt.) = 24 grs.

1 ounce = 20 dwt. = 8 drachms.
1 pound = 12 oz. = 5760 grs.
(1 lb. avoirdupois = 7000 grs.)

Money

1 shilling (s.) = 12 pence (d.).
1 pound (£) = 20s.
Also farthing ($\frac{1}{4}$d.), florin (2s.), half-crown (2s. 6d.) and guinea (21s.).

Volume

(Liquid)

1 gallon = 4 quarts = 8 pints
= 32 gills = ·16054 (1/6·2290) cu. ft.
1 gallon water (62° F.) weighs 10 lb. av.
(in air against brass weights).

---

1 fluid drachm (ʒ) = 60 minims (♏).
1 fluid oz. (℥) = 8 dr. = $\frac{1}{20}$ pint.
1 fluid oz. water (62° F.) weighs 1 oz. av.

(Dry)

1 peck = 2 gals. 1 bushel = 8 gals.
1 sack = 4 bush. 1 quarter = 8 bush.
(1 firkin = 9 gals. 1 barrel = 36 gals.)

---

1 register ton = 100 cu. ft.
1 rod = 1000 cu. ft.

Weights and Measures of the United States of America

Weights and measures are the same as the English systems, except for small differences in standards, and the following exceptions:—

1 hundredweight (short) = 100 lb., and 1 ton (short) = 20 cwt. = 2000 lb. The English (long) ton, 2240 lb., is also used.

Dry measures of volume are 3 per cent. less than the corresponding English measures, being based on the Winchester bushel instead of the Imperial bushel. Fluid measures of volume are $\frac{5}{6}$ of the corresponding English measures. (Hence in the U.S. system 1 bushel = 9·3092 fluid gallons.) Also 1 U.S. fluid ounce = $\frac{1}{16}$ U.S. pint, so that a fluid ounce of water weighs 1·0408 oz. av. (See also conversion factors.)

A township = 36 sq. miles. A billion (French *Milliard*) = 10^9, a trillion = 10^{12}. (English billion = 10^{12}, trillion = 10^{24}.)

The Metric System

The metre is (very nearly) 10^{-7} of the distance from the pole to the equator.

The gram is (very nearly) the mass of 1 cubic cm. of water at maximum density (3·98° C.). (Actual mass of 1 c.c. of water = ·999973 g.)

The litre is (very nearly) 1000 c.c. (1 litre = the volume of 1000 g. of water at 3·98° C. = 1000·027 c.c.)

Also 1 are = 100 sq. metres, so that 1 hectare = 10,000 sq. m. = $\frac{1}{100}$ sq. km.; 1 metric quintal (doppel zentner, U.S.S.R. centner) = 100 kg.; 1 metric ton = 1000 kg.; 1 Continental pound = $\frac{1}{2}$ kg.; 1 stere = 1 cu. m. 1 micron (μ) = 10^{-6} m. 1 angstrom unit (Å) = 10^{-10} m.

Prefixes: micro-, 10^{-6}; mille-, 10^{-3}; centi-, 10^{-2}; deci-, 10^{-1}; Deka-, 10; Hecto-, 10^2; Kilo-, 10^3; Myria-, 10^4; Mega-, 10^6.

Conversion Factors

Basic Factors

1 yard (G.B.) = ·914,399 metres.
1 yard (U.S.A.) = ·914,402 metres.
1 pound = 453·5924 grams.
1 gallon (G.B.) = 4·545,963 litres.
1 gallon (U.S.A.) = 3·785,332 litres.
1 bushel (U.S.A.) = 35·238,33 litres.

Length

1 inch = 2·5400 (1/·39370) cm.
1 foot = ·30480 (1/3·2808) m.
1 yard = ·91440 (1/1·0936) m.
1 mile = 1·6093 (1/·62137) km.

Weight

1 grain = ·064799 (1/15·432) g.
1 ounce = 28·350 (1/·035274) g.
1 pound = ·45359 (1/2·2046) kg.
1 cwt. = ·50802 (1/1·9684) m. quintals.
1 short cwt. = ·45359 (1/2·2046) m.q.
1 ton = 1·0160 (1/·98421) metric tons.
1 short ton = ·90718 (1/1·1023) m.t.

Capacity (Great Britain)

1 pint = ·56825 (1/1·7598) litres.
1 gallon = 4·5460 (1/·21998) litres.
1 bushel = 36·368 (1/·027497) litres.
1 British pint, gallon, etc. = 1·20094 (1/·83268) U.S. fluid pints, gallons, etc.
1 British bushel, etc. = 1·03205 (1/·96895) U.S. bushels, etc. (dry).

Capacity (U.S.A.)

Fluid { 1 pint = ·47317 (1/2·1134) lit.
1 gall. = 3·7853 (1/·26418) lit.
Dry: 1 bush. = 35·238 (1/·028378) lit.

Volume

1 cu. inch = 16·387 (1/·061024) c.c.
1 cu. foot = ·028317 (1/35·315) cu. m.
1 cu. yard = ·76455 (1/1·3080) cu.m.

Area, etc.

1 sq. inch = 6·4516 (1/·15500) sq. cm.
1 sq. foot = ·092903 (1/10·764) sq. m.
1 sq. yard = ·83613 (1/1·1960) sq. m.
1 sq. mile = 2·5900 (1/·38610) sq. km.

1 acre = ·40468 (1/2·4711) hectares.
1 lb. per acre = 1·1209 (1/·89218) kg. per ha.
1 cwt. per acre = 1·2554 (1/·79659) m.q. per ha.

1 ton per acre = 2·5107 (1/·39829) m. tons per ha.
1 bushel per acre = 89·867 (1/·011128) litres per ha. (U.S.A.) = 87·076 (1/·011484) litres per ha.

Bushel weights (standard): wheat, 63 lb.; oats, 42 lb.; barley, 56 lb.; maize, 60 lb.; beans, peas, 64 lb.

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